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Crosscutting Areas

Selling Passes to Strategic Customers

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Abstract. Passes are prepaid packages of multiple units of goods or services with flexible consumption times. They may take a variety of forms such as commuter passes in transportation, capped quotas in telecommunications, or memberships in health or beauty clubs. We consider a monopolist selling products or services to strategic customers by dynamically pricing passes in conjunction with individual items. The strategic behavior is captured by a dynamic choice model that endogenizes strategic purchase, utilization, and renewal of the pass. Under the control-theoretic framework, we find that the optimal pricing policy has a turnpike property; the optimal price trajectories stay near the steady state for most of the sales horizon, and the fixed-pricing policy performs very well when the horizon is long enough. In the turnpike, we show that passes should offer a quantity discount when customers are not fully strategic. As a form of advance purchase, passes allow the firm to capitalize on strategic behavior without limiting the supply. From the revenue-recognition principle, we show that a passholder can generate a higher revenue rate than a nonpassholder customer.

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1. Introduction

Most customers of theme parks, transportation, and gyms are now familiar with *passes*, which consist of a prepaid package of (possibly unlimited) credits that can be redeemed for future use before an expiration date. Public transportation has a long history of selling prepaid bundles of tickets in the form of stored-value cards or tokens. Many recreational venues such as ski resorts offer seasonal passes that grant the holder multiple access for one season without additional charges. Passes are also sold under various names or forms. For example, the instant messaging app Skype sells a fixed number of "Skype credits" that can be redeemed toward international texting or calling to mobile phones; this is essentially a type of pass. Many prepaid vouchers, health or beauty club memberships, or meal plans are also examples of passes.

Passes are commonly sold and managed through the same sales channel as individual items, so customers can choose between an individual purchase and the pass. In addition to the convenience and discount over individual purchase, passes may also offer additional perks such as priority service, free parking, or coupons exclusive to passholders.

Dynamic pricing is becoming more prevalent in various industries. In dynamic pricing settings, passes offer various benefits to the customers. For example, they can lock in prices and protect passholders against future price changes. They also allow passholders to strategize on the utilization of passes and maximize their utilities. That is, a strategic passholder may use the pass when the price of an individual item is high but buy individually when the individual price is low. However, this is not without risk because passholders may fail to use up all the credits on the pass by the expiration date. Therefore, they need to carefully evaluate the utility of each credit on the pass. Note that the utility of a credit is entangled with future individual prices. For example, if future individual prices were higher, then the utility of a credit would also be higher because the customer could use it to circumvent higher prices and achieve more savings.

Passes also present considerable challenges for the firm's pricing decisions. First, passes naturally divide customers into two segments—passholders and nonpassholder customers—with possibly different behaviors. Passholders can redeem their credits in exchange for services or products, but such an option is obviously not available to nonpassholder customers. Furthermore, passholders can be heterogeneous; empirical studies show that the redemption rate depends on the remaining credit balance (Andrews et al. 2014): customers tend to consume more when the remaining balance is sufficient but curb the usage when the balance becomes low (Assuncao and Meyer 1993, Folkes et al. 1993). This suggests that the passholder segment may be further divided into many subsegments with different credit balances.

Second, customers continuously migrate between the segments. A nonpassholder customer becomes acpassholder by purchasing the pass. But when the credits on the pass are used up, he or she becomes a nonpassholder again. In addition, passholders can migrate from one credit level to another by redeeming credits. Note that these migrations are driven by customers' own choices, such as pass purchase, credit redemption, or wait. Because these choices are price sensitive, the migrations can be influenced by the firm's pricing decisions.

The firm can potentially price passes dynamically but must coordinate this with the dynamic individual prices. To jointly price passes and individual items in a dynamic way, the firm needs to carefully account for customer choices and the resulting migrations. For example, passes may cannibalize the profit from individual items when a customer who is willing to pay higher individual prices ends up paying less with the pass. A pass sale now may also cannibalize future sales because the passholders need not pay when they redeem the credits in the future. A dynamic pass price also influences strategic credit redemptions because passholders may time their redemption to renew their passes at low prices (Ailawadi and Neslin 1998, Bell et al. 2002). Additionally, as mentioned earlier, the redemption decision depends on individual price because the credits can protect passholders against individual price changes.

In this paper, we develop a dynamic model for a monopoly to jointly price passes and individual items in the face of strategic customers. Customers optimize their sequential purchase and redemption decisions to maximize their intertemporal utilities, so the profit-maximizing firm must account for customers' strategic decisions when it sets the prices. In our model, customer choice and the corresponding migrations are all endogenized: we explicitly model how customers make strategic purchase and redemption decisions (such as what to buy, when to buy, and when to redeem) and how they migrate from one segment (or subsegment) to another. The model is stylized but incorporates all aforementioned attributes and challenges that are important in selling passes. We use this model to address the following questions: How should the firm price passes in conjunction with individual items? How do price, demand, and profit depend on the forward-looking behaviors of strategic customers? And what are the benefits of selling passes, if any?

We propose a control-theoretic modeling framework that endogenizes customers' rational sequential choices in a dynamic pricing setting. It integrates the dynamic choice models with the optimal control theory, allowing us to study the optimal pricing problem involving complex strategic behavior. The framework is based on a fluid model operating in continuous time, which is different from the game-theoretic framework (Besanko and Winston 1990, Levin et al. 2009).

We observe that the optimal pricing policy has a *turnpike* property: the optimal price trajectories stay close to the steady state for most of the time except near the beginning and the end of the horizon, provided that the sales horizon is sufficiently long (to the extent that passholders have plenty of time to spend all their credits). In this sense, the trajectory of the optimal price is similar to the best route of driving from one city to another city: when the two cities are far apart, it is best to take the turnpike and stay there for most of the time until one comes close to the destination.

The presence of the turnpike property suggests that the firm should adopt a "lazy" pricing policy. During the turnpike regime, it should avoid frequent price changes to mitigate complex dynamic responses from strategic customers. Although complex dynamic behaviors may still take place, they are largely confined to the beginning and end of the sales horizon. In fact, the longer the sales horizon is, the longer the prices stay near the steady state, and a greater percentage of profit is generated near the steady state.

Near the turnpike, passes should offer fewer quantity discounts when customers are more forward looking. An interesting finding is that customer's forward-looking behaviors can boost the total profit of the firm offering passes (even under ample supply). This finding contradicts the general intuition that forward-looking behaviors are detrimental to profit. Passes use the advance purchase to capitalize on forward-looking behavior by exploiting customer uncertainty about future utilities (Shugan and Xie 2000).

Because passholders prepay for their future consumption, a natural question is, does a passholder generate the same profit rate as a nonpassholder customer eventually? If yes, then passes simply shift the future profit to the present. However, we show that in most practice-relevant settings, a passholder generates a higher profit rate than a nonpassholder customer and thus is more valuable to the firm. We further identify key drivers of the excess profit of passes.

2. Relevant Literature

the literature on pass pricing in the presence of strategic customers is limited. Here we highlight some areas related to different facets of the problem.

2.1. Advance Purchase

Because the pass requires customers to prepay for future consumption, it can be viewed as a form of advance purchase. The customers pay for the right of future consumption, which may or may not lead to actual consumption. It is already understood that advance purchase has several benefits to the firm: it can mitigate the firm's information disadvantages by shifting some uncertainties to the customers (Shugan and Xie 2000) and hence generate higher profit when customers are uncertain about their future valuations (Dana 1998, Xie and Shugan 2001). If some credits remain unused at the expiration date, the firm can view the related sales as profit (also known as *breakage* in accounting).

2.2. Multiunit Pricing

Advance purchase is just one facet of the problem. What the pass advance sells is a package of multiple identical items. In reality, passes almost always offer some quantity discounts, where the unit price decreases with the purchase quantity. The literature on quantity discounts is rich in economics and marketing (Wilson 1993). It is known that quantity discounts can price discriminate customers with different consumption volumes to improve revenue (Dolan 1987). However, most of the quantity discount models in the existing literature are static and hence are different from our dynamic model. In addition, our model considers customers choosing between individual items and the pass, so it is naturally related to multiunit pricing problems with choices (Maskin and Riley 1984, Tirole 1988, Stole 2007). A common assumption in that literature is that customers are not forward looking, but in our context, the utility of a pass also depends on the future dynamic prices that the credits can circumvent. It also depends on time because the utility diminishes near expiration. These dynamics are not captured by existing models.

Although a pass combines advance purchase and multiunit sales, the joint dynamic pricing problem is more than a straightforward integration of existing advance purchase and multiunit pricing problems because the firm needs to consider pass-related choices made by the customers, including purchases, redemptions, and renewals. For example, the credit redemption process is inherently dynamic and strategic, whereas the primary focus of the existing literature is on only the initial, not subsequent, purchases. Empirical studies show that the redemption rate depends on the remaining credit balance as well as on the time to expiration (Andrews et al. 2014). With sufficient credits in hand, customers tend to consume at a faster rate (Assuncao and Meyer 1993). The consumption rate will decrease with consumption, because a lower credit balance is perceived as more valuable (Folkes et al. 1993).

Moreover, the sale of a package of multiple units can create a temporary drop in future demand for individual items because the passholders do not pay when they redeem the credits. This is similar to the *postpromotion dip* caused by strategic stockpiling (Macé and Neslin 2004, Su 2010). This temporal cannibalization of future demand is another difference between pass pricing and existing multiunit pricing models.

2.3. Bucket Pricing

The pass sets a flat fee for a maximum allowance for consumption, or quota, which is similar to bucket pricing in telecommunications industries (Sun et al. 2006, Schlereth and Skiera 2012). The literature on bucket pricing is dominated by empirical studies that examine customer choices among different plans (i.e., price and quota combinations) rather than the optimal pricing decisions. Sun et al. (2006) studied customer behaviors in online movie rentals and found that customers often overpay for their actual consumption. This underutilization allows the firm to "oversell" and capitalize on the unused items. The time horizon for bucket pricing programs is typically short; as a result, most customers cannot use up the quota before it expires. By contrast, the pass is usually valid for a sufficiently long time and allows passholders to spend all their credits and subsequently renew the pass. In this sense, pass pricing is probably more intricate than bucket pricing. It is interesting to note that Sun et al. (2006) also observe that customer behavior indeed depends on long-term perception of cost, as we speculated earlier. We endogenize this dependence in our model. At a high level, our work is also related to the literature involving forward-looking customers; see Chen and Shi (2017) and Chen and Chu (2018) for recent developments.

2.4. Unlimited Passes

We focus on the multipass or the limited pass that contains a finite number of credits, which is more difficult to analyze than the unlimited pass that contains an infinite number of credits. Carbajo (1988) has studied the static pricing problem for unlimited passes with myopic customers, in which the unlimited pass is viewed as a special case of two-part tariff (Oi 1971). The pass price can be considered as a lump-sum fee, and the per-unit charge is 0. Because of the zero per-unit charge, a customer with an unlimited pass has no incentive to buy more items individually. In addition, passholders can never migrate to the nonpassholder segment by themselves until the pass expires. With limited passes, however, customers may eventually run out of the pass credits and loose the price protection; passholders need to be skillful at redemption. In the same vein, the firm must account for strategic redemption, repeated pass purchases, and bidirectional migrations between two segments. Consequently, pricing the limited pass is more complex than pricing the unlimited pass.

3. The Joint Pricing Model

The pricing problem takes place continuously over a finite sales horizon $[0, T]$, and there is an abundant supply. We apply the optimal control theory to derive the open-loop pricing policy for the firm (Sethi and Thompson 2000). The firm announces the entire price trajectories at the beginning of the sales horizon and commits to the prices. We will discuss price commitment after Theorem 1. It can also be justified in many practical settings; see Chen and Shi (2017) and the references therein. We consider the deterministic open-loop control problem for the sake of tractability, yet this framework can reveal useful insights into the corresponding stochastic problem and help in finding good heuristics (a detailed discussion is given in Section EC.26 of the online appendix).

It is mathematically convenient to use a fluid model to approximate the sales process. That is, each item is treated as infinitesimal, and sales unfold continuously. There is a finite population of customers; each customer is again infinitesimal, and the whole population is normalized to 1. Random decision opportunities arise at a rate of λ, which can be interpreted as the maximum shopping opportunities. A customer may or may not purchase when a shopping opportunity arises, and thus the actual demand rate depends on the customer's choice probabilities and can vary over time.

An item may be sold individually or with the redemption of one credit by a passholder. The firm decides both the price of individual items f_t and the price of the pass p_t for all $t \in [0, T]$. Each pass includes $\bar{k}+1$ credits initially. Because pass utilization typically commences very quickly after purchase, we assume that the first credit is used immediately, and the passholder has $\bar{k}$ credits to use in the future. A customer can hold at most one pass at a time and does not stockpile multiple items or profit from arbitrage by reselling his or her own pass credits to other customers. A passholder remains a potential source of renewing the pass later when he or she spends all the credits on the pass. All unused credits expire at the end of the horizon.

Customers are forward looking and maximize their expected utilities by strategically choosing among multiple alternative actions based on various personal factors that may not be observable to the firm. More important, customers themselves face substantial *uncertainty* about their future consumption states when purchasing the pass because the purchase time is separated from the consumption time. Therefore, the behavior of each individual customer appears to be random to some extent. However, the firm can calculate the probability that a specific alternative is chosen, and consequently, the customer population as a whole evolves in a deterministic manner because each customer is infinitesimal.

Consider a nonpassholder customer facing a decision opportunity at time t. With no pass credit in hand, he or she can choose an individual item, choose the pass, or do nothing. Let π^s_{0t} denote the probability of purchasing an individual item, in which case nonpassholder customer pays the individual price f_t and remains a nonpassholder customer with no credit. Similarly, if he or she did nothing, he or she also remains a nonpassholder customer until the next decision opportunity when he or she can decide again. But if he or she purchased the pass (which occurs with probability π^p_{0t}), he or she pays the pass price p_t, uses the first credit immediately, and keeps the rest of the $\bar{k}$ credits for future use. By doing so, he or she becomes a passholder and can strategize on credit redemption. Choices available to him or her include the redemption of a credit, the purchase of an individual item, and doing nothing. Given a decision opportunity at time t, a passholder with k remaining credits purchases an individual item with probability π^s_{kt} and still keeps k credits in hand. He or she can also redeem a credit with probability π^r_{kt}, after which he or she has $k-1$ credits remaining. Similar decisions are repeated at subsequent random decision opportunities until he or she depletes credits on the pass and becomes a nonpassholder customer again, from which point he or she continues the decision-making process. The cycle of decisions is illustrated in Figure 1.

Because the entire customer population is normalized to 1, we use w_{kt} to represent the population of customers with k credits at time t, where $k = 0, \ldots, \bar{k}$, and w_{0t} represents the population of nonpassholder customers. The rate λ is split among customers with different credit balances in proportion to w_{kt}. From an individual customer standpoint, the resulting purchase or consumption processes are nonhomogeneous compound Poisson processes. Intensities of individual and pass purchases by nonpassholder customers are $\lambda w_{0t}\pi^s_{0t}$ and $\lambda w_{0t}\pi^p_{0t}$, respectively. Similarly, $\lambda w_{kt}\pi^s_{kt}$ and $\lambda w_{kt}\pi^r_{kt}$ are the intensities of individual and credit redemption by passholders with k remaining credits, respectively, where $k = 1, \ldots, \bar{k}$. Customers migrate among different credit levels according to a recurrent continuous-time Markov chain. The probability distribution of this chain corresponds to the population distribution, and its continuous-time evolution is described by the following ordinary differential equations:

$$\begin{aligned}\dot{w}_{kt} &= \lambda w_{(k+1)t}\pi^r_{(k+1)t} - \lambda w_{kt}\pi^r_{kt}, \qquad k = 1, \dots, \bar{k}-1,\\ \dot{w}_{\bar{k}t} &= \lambda w_{0t}\pi^p_{0t} - \lambda w_{\bar{k}t}\pi^r_{\bar{k}t},\\ \dot{w}_{0t} &= \lambda w_{1t}\pi^r_{1t} - \lambda w_{0t}\pi^p_{0},\end{aligned} \tag{1}$$

with initial condition $w_{00} = 1$ and $w_{k0} = 0$, for $k = 1, \dots, \bar{k}$. These differential equations can be easily interpreted by noting that the net rate of change in each subpopulation is the difference between the inflow rate and the outflow rate. The initial conditions assume that there are no passholders at the beginning of the sales horizon.

Up to this point, we have treated the choice probabilities $\pi^p_{0t}, \pi^s_{0t}, \pi^r_{kt}$ and π^s_{kt} as given. In fact, they all depend on price trajectories, the number of remaining credits, time to expiration, and customers' forward-looking behaviors. We model how a strategic customer population responds to dynamic prices using the dynamic choice model (Rust 1987), which is well suited for situations where customers make their sequential decisions to maximize the expected discounted utilities. Details about this model and the related derivations are found in Section EC.1 of the online appendix, in which we first model how individual customers evaluate the choice alternatives over time and then aggregate the choices from the entire population. The resulting dynamic choice model is presented in the following theorem.

Theorem 1 (Customer's Dynamic Choice Model). *Let U_{kt} denote the customer's expected utility of holding k credits at time t. Under Assumptions* EC.1–EC.3 *in the online appendix, $\{U_{kt}, k = 0, \dots, \bar{k}\}$ satisfy the following ordinary differential equations:*

$$\begin{aligned}\dot{U}_{kt} &= -\lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - U_{kt} + U_{(k-1)t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t}{\mu\gamma}\right)\right]^{\gamma}\right\} + \rho U_{kt},\ k = 1, \dots, \bar{k},\\ \dot{U}_{0t} &= -\lambda\mu \ln\left\{1 + \left[\exp\left(\frac{a - p_t + U_{\bar{k}t} - U_{0t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t}{\mu\gamma}\right)\right]^{\gamma}\right\} + \rho U_{0t},\end{aligned} \tag{2}$$

with terminal conditions $U_{kT} = 0$. Furthermore, the solution to this terminal value problem is unique for any given pricing policy,[1] *and the corresponding choice probabilities are given by*

$$\pi^p_{0t} = \frac{\exp\left(\frac{a - p_t + U_{\bar{k}t}}{\mu\gamma}\right)\cdot\left[\exp\left(\frac{a - p_t + U_{\bar{k}t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{0t}}{\mu}\right) + \left[\exp\left(\frac{a - p_t + U_{\bar{k}t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, \tag{3}$$

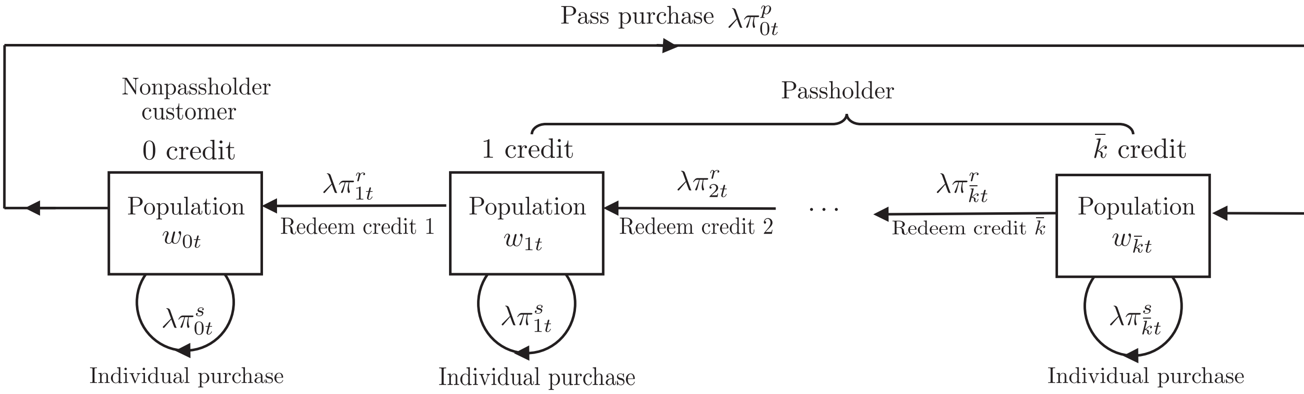
$$\pi^s_{0t} = \frac{\exp\left(\frac{a - f_t + U_{0t}}{\mu\gamma}\right)\cdot\left[\exp\left(\frac{a - p_t + U_{\bar{k}t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{0t}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{0t}}{\mu}\right) + \left[\exp\left(\frac{a - p_t + U_{\bar{k}t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{0t}}{\mu\gamma}\right)\right]^{\gamma}}, \tag{4}$$

$$\pi^r_{kt} = \frac{\exp\left(\frac{a + U_{(k-1)t}}{\mu\gamma}\right)\cdot\left[\exp\left(\frac{a + U_{(k-1)t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{kt}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{kt}}{\mu}\right) + \left[\exp\left(\frac{a + U_{(k-1)t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{kt}}{\mu\gamma}\right)\right]^{\gamma}}, \tag{5}$$

$$\pi^s_{kt} = \frac{\exp\left(\frac{a - f_t + U_{kt}}{\mu\gamma}\right)\cdot\left[\exp\left(\frac{a + U_{(k-1)t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{kt}}{\mu\gamma}\right)\right]^{\gamma-1}}{\exp\left(\frac{U_{kt}}{\mu}\right) + \left[\exp\left(\frac{a + U_{(k-1)t}}{\mu\gamma}\right) + \exp\left(\frac{a - f_t + U_{kt}}{\mu\gamma}\right)\right]^{\gamma}},\quad k = 1, \dots, \bar{k}. \tag{6}$$

All proofs in this paper are located in the online appendix. The demand model described in this theorem takes the price paths (f_t, p_t) as inputs and produces the expected utilities (U_{kt}) and then choice probability paths $(\pi^p_{0t}, \pi^s_{0t}, \pi^r_{kt}, \pi^s_{kt})$ as deterministic outputs. Because the credit k can be redeemed at any time before T, the corresponding expected utility U_{kt} should depend on the entire price trajectories from t to T. It also should depend on the time to expiration because a credit has lower value when it is expiring soon. These effects are indeed captured by the model (see Section EC.3 in the online appendix). Note that the terminal conditions $U_{kT} = 0$ suggest that any unused credits have no value at the end of the horizon. The recursion in (2) starts from a credit's expiration time and works backward in time to value each credit by backward induction. As a result, the expected utility U_{kt} depends on all future price paths[2] $\{f_s, p_s, t \le s \le T\}$,

Figure 1. A Customer's Decision and Migration Cycle



the number of remaining credits k, and the time to expiration $T - t$. It is also important to note that U_{kt} does not depend on past price paths, which ensures that the optimal pricing policy is time consistent.

The discount factor ρ is a measure of the level of strategic behavior; a lower ρ is associated with more strategic customers. When $\rho \to 0$, the customers are *fully strategic*. When $\rho \to \infty$, the customers are *myopic*. For $0 < \rho < \infty$, the customers are *partially strategic*, which is a more realistic case. It is important to note that ρ influences the choice probabilities by influencing the expected utilities U_{kt} through differential Equations (2). Similarly, future price paths, the number of remaining credits, and the time to expiration all influence the current choices.

The choice probabilities take the form of the nested logit model, in which the parameter $\gamma \in (0, 1]$ measures the degree of independence between the idiosyncratic components of utilities of individual consumption and the consumption with a pass, with a larger value representing greater independence; γ can also be viewed as a measure of the unobserved heterogeneity in customer preferences. The largest heterogeneity occurs when $\gamma = 1$, in which case our model reduces to the dynamic logit model (Rust 1994). Although the choice probabilities for this case take the logit form, they do *not* exhibit the *independence from irrelevant alternatives* (IIA) property because the expected utilities U_{kt} depend on the attributes of irrelevant alternatives.[3] Moreover, although the dynamic logit model is originally derived from the random utility theory, it also arises from the optimal information-processing policy in an information-theoretic model of rational inattention (Steiner et al. 2017). In the rational inattention theory, customers make optimal decisions on processing costly information about the choice alternatives (Sims 2003).

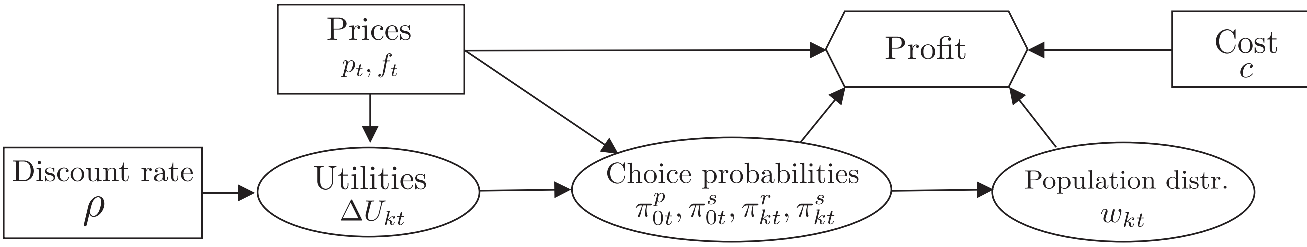
The other extreme case, $\gamma \to 0$, represents a situation where the idiosyncratic components of utilities of individual and pass consumption are perfectly correlated. This situation is uncommon in practice because the pass is naturally differentiated from the individual ticket. Specifically, the pass is based on advance selling of multiple tickets, and its value depends on the customer's consumption state, which can vary from time to time and from customer to customer. For example, the value of a multiday ski pass may depend on the customer's state, including health and mood, scheduling conflicts, state of companions, projects at work, and weather condition. Customers may enjoy the pass more when they are energetic and not preoccupied but like it less when they are fatigued or have uncompleted projects at work. Even if the pass provides the same service as the individual ticket, customers' preferences may still diverge as a result of heterogeneity in the estimation of future consumption states. Budget constraints also contribute to the differentiation; because pass buyers need to pay a lump-sum fee, the customers on a budget may prefer individual purchase. The differentiation is further enhanced by the convenience and additional perks associated with the pass.

Finally, we can see in Theorem 1 that the choice probabilities depend on U_{kt} only through their differences. Thus, we introduce a new state variable $\Delta U_{kt} \triangleq U_{kt} - U_{(k-1)t}$, $k = 1, \ldots, \bar{k}$, representing the marginal expected utility of credit k, and we will use ΔU_{kt} going forward. Accordingly, the utility equations (2) can be rewritten as

$$\begin{aligned}\Delta\dot{U}_{kt} &= -\lambda\mu\Big[\ln\Big(1 - \pi^r_{(k-1)t} - \pi^s_{(k-1)t}\Big) - \ln\big(1 - \pi^r_{kt} - \pi^s_{kt}\big)\Big] \\ &\quad + \rho\Delta U_{kt},\ k = 2, \ldots, \bar{k}, \\ \Delta\dot{U}_{1t} &= -\lambda\mu\big[\ln\big(1 - \pi^p_{0t} - \pi^s_{0t}\big) - \ln\big(1 - \pi^r_{1t} - \pi^s_{1t}\big)\big] + \rho\Delta U_{1t}.\end{aligned} \tag{7}$$

The firm incurs a marginal cost c of providing each product or service. The objective of the firm is to

Figure 2. Influence Diagram of the Pricing Decision Problem



maximize the total profit by controlling the individual and pass prices over the sales horizon:[4]

$$\max_{\{p_t, f_t\}} \int_0^T \Bigg[\underbrace{\lambda(p_t - c) w_{0t} \pi^p_{0t}}_{\text{pass sales profit}} + \underbrace{\sum_{k=0}^{\bar{k}} \lambda(f_t - c) w_{kt} \pi^s_{kt}}_{\text{individual sales profit}} - \underbrace{\sum_{k=1}^{\bar{k}} \lambda c w_{kt} \pi^r_{kt}}_{\text{redemption cost}} \Bigg] dt,$$

subject to the state equations and boundary conditions for customer migrations (1), expected utilities (2) or its equivalent (7), and the choice probabilities (3)–(6). The influence diagram of the decision problem is in Figure 2, in which prices influence utilities, and they jointly determine the choice probabilities, which drive the customer migration and determine the population distribution over different credit levels. The profit depends on prices, cost, choice probabilities, and population distribution. We apply Pontryagin's maximum principle to derive the necessary conditions for the optimal pricing policy (Sethi and Thompson 2000). The optimal control policy is characterized by a nonlinear differential-algebraic system of equations given in Section EC.7 of the online appendix.

4. Turnpike Properties

Through extensive numerical experiments, we observe that the optimal pricing policy exhibits a turnpike property: the optimal state and control trajectories stay close to the steady state for most of the sales horizon if

Figure 3. (Color online) An Illustration of Turnpike Properties ($\bar{k} = 2$)

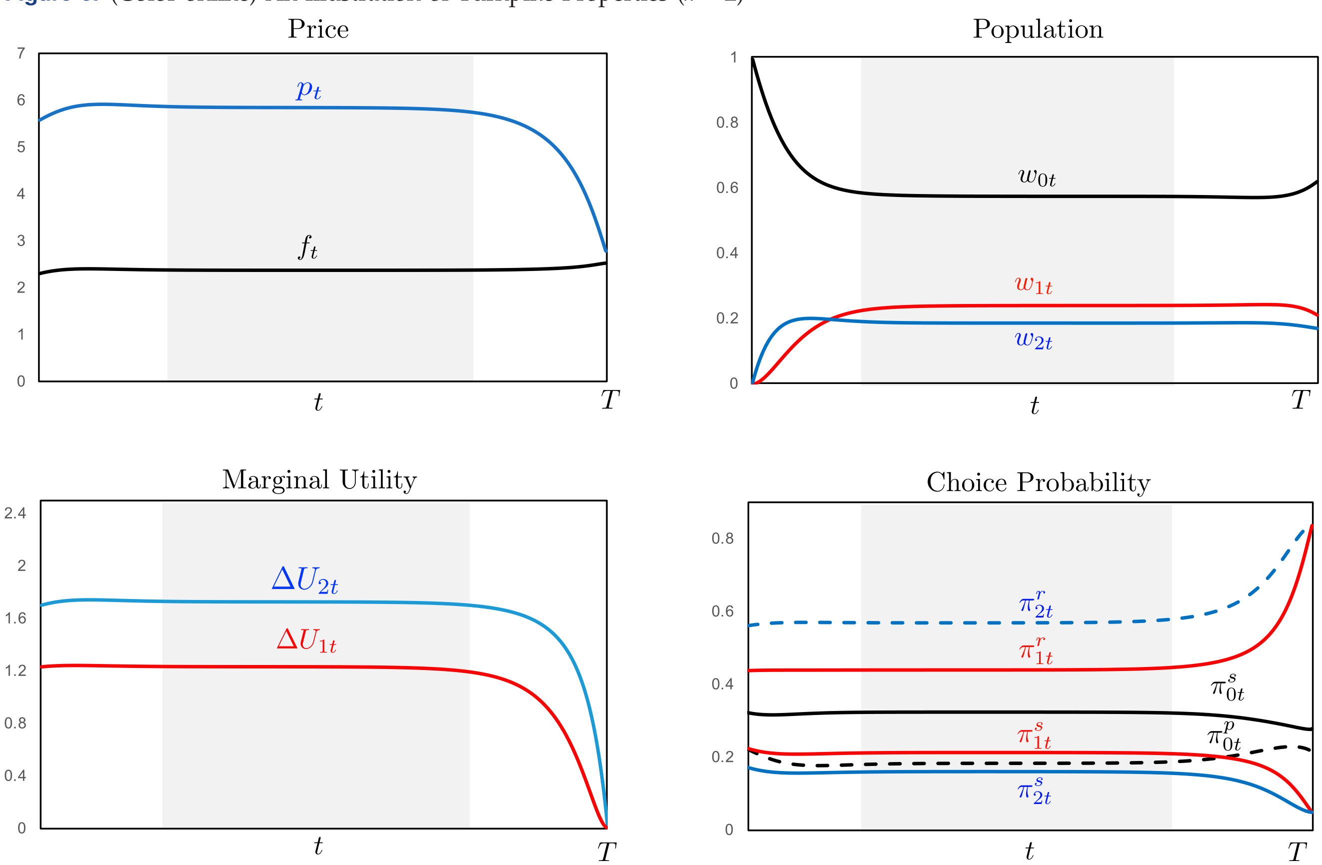
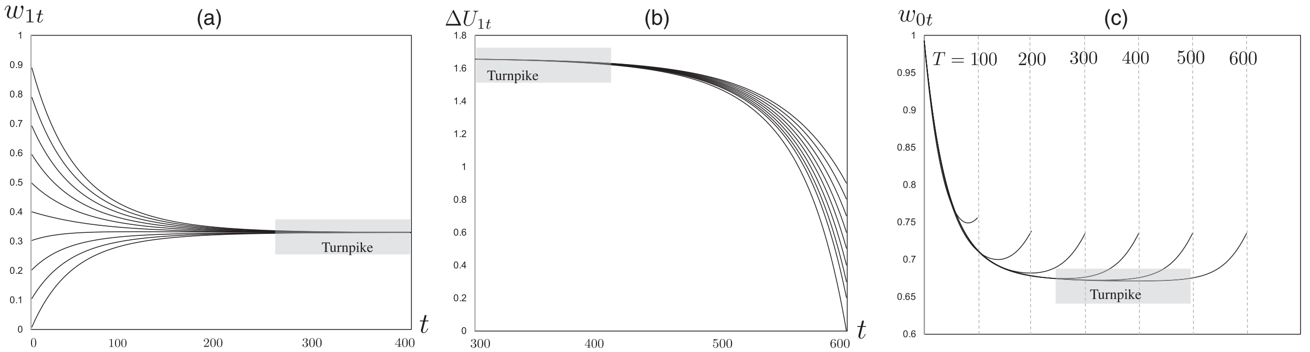


Figure 4. Optimal State Paths with Different (a) Initial Values, (b) Terminal Values, and (c) Horizon Lengths



the horizon is sufficiently long. More specifically, if the sales horizon is longer than a certain threshold such that the steady state can be approached, the optimal trajectories will converge to the turnpike regardless of the initial and terminal conditions. This behavior is similar to driving along a major highway: if one wishes to drive from one city to another across a long distance, then one should first get on a major highway, spend most of the time on it (similar to a steady state), and finally, leave the highway to reach the destination.

An example illustrating the temporal behavior of the optimal price is shown in Figure 3. The parameters are specified as $(T, c, \bar{k}, \lambda, a, \mu, \rho, \gamma) = (1{,}000, 0.1, 2, 0.03, 2, 1, 0.006, 0.9)$. The optimal trajectories can be roughly divided into three stages: an initial adjustment stage, a steady-state stage, and a terminal adjustment stage. The initial adjustment is similar to the turnpike entrance ramp, which transfers the initial condition to the steady state. The second stage lies in the middle of the sales horizon and is the steady state (as shown in the shaded region), during which all state and control variables stay near constant levels. On reaching the terminal adjustment stage, state and control variables start to deviate from the steady state and make continuous adjustments to reach the terminal value (similar to a turnpike exit ramp). Strictly speaking, the optimal path stays close to—but not exactly on—the steady state.

We vary the initial credit distribution w_{10} in Figure 4(a) and the terminal marginal utility ΔU_{1T} in Figure 4(b) to examine how the optimal state path depends on the boundary conditions. Other parameters are chosen as $(T, c, \bar{k}, \lambda, a, \mu, \rho, \gamma) = (600, 0.1, 1, 0.03, 2, 1, 0.006, 0.9)$. We observe that the optimal trajectories associated with different boundary values converge to the same steady state and stay near it in the middle of the horizon. Figure 4(c) illustrates how w_{0t} changes when the horizon length T varies. When the horizon is short (e.g., $T < 300$), the steady state may not be approached. However, as long as its length exceeds a certain level (e.g., $T > 400$), the steady state can always be approached, and in addition, the terminal adjustment stages always appear the same. In practice, the sales horizons are generally long enough to make the turnpike reachable. This is so because the steady state represents a scenario where most passholders can spend all the credits on the pass and then renew their passes. A horizon that is too short for full utilization may ruin the firm's reputation or violate regulations.

The presence of a turnpike property might appear surprising in our context. As discussed in Section 1, the strategic behavior of passholders can be very complex in a dynamic pricing situation. The turnpike property suggests that this complexity is confined to only the initial and terminal stages of the sales horizon. In most times, the firm should keep prices stable to mitigate the complex strategic behaviors. It is also worth noting that even when facing constant prices, strategic customers may still delay purchase and wait until the

Table 1. The Performance of the Fixed Price Approximation

Arrival rate	Horizon	$T = 100$	$T = 200$	$T = 300$	$T = 400$	$T = 500$	$T = 600$	$T = 700$	$T = 800$	$T = 900$
$\lambda = 0.01$	Optimal profit	1.216	2.341	3.437	4.524	5.606	6.687	7.767	8.846	9.926
	Approx. profit	1.174	2.297	3.394	4.480	5.563	6.643	7.724	8.803	9.883
	Approx. error	0.042	0.045	0.044	0.043	0.043	0.043	0.043	0.043	0.043
	Relative error (%)	3.42	1.90	1.28	0.96	0.77	0.64	0.55	0.49	0.43
$\lambda = 0.04$	Optimal profit	4.749	9.287	13.819	18.350	22.881	27.411	31.942	36.473	41.004
	Approx. profit	4.702	9.242	13.773	18.304	22.835	27.366	31.897	36.428	40.959
	Approx. error	0.048	0.045	0.045	0.045	0.045	0.045	0.045	0.045	0.045
	Relative error (%)	1.00	0.49	0.33	0.25	0.20	0.17	0.14	0.12	0.11

valuation of consumption becomes higher (namely, wait until they need it the most), as strategic customers aim to maximize their expected surplus.

The turnpike is characterized by a system of algebraic equations given in Section EC.9 of the online appendix. It consists of the steady-state solutions to the state and adjoint equations, as well as the first-order conditions for the Hamiltonian. Let $\bar{w}_k, \Delta\bar{U}_k$ denote the turnpike state variables, and let $\bar{\eta}_k^w, \bar{\eta}_k^u$ denote the adjoint variables. The steady-state prices are denoted by $\bar{f}, \bar{p}$, and choice probabilities are denoted by $\bar{\pi}_0^p, \bar{\pi}_0^s, \bar{\pi}_k^r, \bar{\pi}_k^s$.

The existence of the turnpike property naturally gives rise to the idea of using steady-state prices to approximate the dynamic prices. In Table 1, we compare the profit from the optimal dynamic prices with that from the turnpike prices for different lengths of sales horizon and arrival rates. The parameters for this example are $(\bar{k}, a, \mu, \gamma, c) = (1, 2, 1, 0.3, 0.1)$. The relative approximation error diminishes as the sales horizon becomes longer. This suggests that selling both passes and individual items at flat rates may be close to optimal. In addition, the flat rates perform better when the decision opportunities arise at a higher rate.

We now focus on what happens in the turnpike. For brevity, we may omit the term *turnpike* from this point on. To begin with, we show that each credit has a positive marginal value.

Lemma 1. *The marginal utility is nonnegative in the turnpike (i.e., $\Delta\bar{U}_k \geq 0$) for $k = 1, \ldots, \bar{k}$.*

The following proposition shows that passes can result in heterogeneous consumption probabilities and that the heterogeneity is mediated by the level of strategic behavior.

Proposition 1 (Heterogeneous Consumption Rate). *The turnpike choice probabilities are ordered as $\bar{\pi}_0^s \geq \bar{\pi}_1^s \geq \cdots \geq \bar{\pi}_{\bar{k}}^s$, $\bar{\pi}_0^p \leq \bar{\pi}_1^r \leq \cdots \leq \bar{\pi}_{\bar{k}}^r$, and $\bar{\pi}_0^s + \bar{\pi}_0^p \leq \bar{\pi}_1^s + \bar{\pi}_1^r \leq \cdots \leq \bar{\pi}_{\bar{k}}^r + \bar{\pi}_{\bar{k}}^s$. The equalities are obtained if and only if $\rho = 0$.*

This proposition suggests that when customers are not fully strategic ($\rho > 0$), a customer with more credits is less likely to buy an individual item and more likely to redeem the credit. Moreover, credits boost consumption: the combined consumption probability (through the pass or individual purchase) is highest when the pass is newly purchased, and it will decrease with credit utilization, reaching the lowest point when all credits are spent. However, in the extreme case where customers are fully strategic ($\rho = 0$), the number of remaining credits has no impact on the turnpike choice probabilities. This is so because, in the turnpike, fully strategic customers value and use the pass credit in the same way as an individual ticket.

Proposition 2 (Diminishing Credit Utility).

a. *The marginal turnpike utilities are ordered as $\Delta\bar{U}_1 \geq \Delta\bar{U}_2 \geq \cdots \geq \Delta\bar{U}_{\bar{k}}$.*

b. *The turnpike pass price is bounded by $\bar{p} \geq \Delta\bar{U}_1 + \sum_{k=1}^{\bar{k}} \Delta\bar{U}_k$.*

Proposition 2 suggests that the marginal utilities of credits are diminishing in the credit balance. Customers place a higher value on an item that is scarce. Furthermore, the pass price is higher than the marginal utilities aggregated from all credits.

Let $\Delta\bar{\eta}_k^w$ denote the turnpike shadow price of the kth credit on the pass (see Section EC.7 of the online appendix for details). We now introduce the notion of *economic profit* to be used throughout this paper. It is defined as the price net of all the corresponding shadow prices and cost. For example, the economic profit of a pass is $\bar{p} - \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w - c$—namely, the pass price minus the shadow prices of all credits and the cost of serving the first use. Similarly, the economic profit of credit k is $\Delta\bar{\eta}_k^w - c$—namely, the shadow price of that credit net of the cost of serving a redemption. The economic profit of an individual item is always $\bar{f} - c$, the individual price net of the marginal cost. To calculate the *expected* economic profit, we multiply the economic profits by the corresponding probabilities.

Proposition 3 (Constant Expected Economic Profit). *The expected economic profit is invariant with respect to (w.r.t.) the number of credits remaining. That is,*

$$\alpha \triangleq \bar{\pi}_0^p\left(\bar{p} - \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w - c\right) + \bar{\pi}_0^s(\bar{f} - c)$$
$$= \bar{\pi}_k^r(\Delta\bar{\eta}_k^w - c) + \bar{\pi}_k^s(\bar{f} - c)$$

for $k = 1, \ldots, \bar{k}$.

This proposition suggests that every customer, regardless of credit balance, generates the same expected economic profit in the turnpike. It is important to distinguish expected *economic* profit from expected profit, which is not necessarily invariant with respect to the credit balance (see Section 7).

5. Fully Strategic/Myopic Customers

The closed-form solution to the general turnpike equations in Section EC.9 of the online appendix can be extremely difficult, if not possible, to obtain. To gain insights into the pricing policy, we begin with the following two special cases: fully strategic customers ($\rho = 0$) and myopic customers ($\rho \to \infty$).

Proposition 4 (Fully Strategic Customers). *For fully strategic customers ($\rho = 0$), the solution to the turnpike equations is unique and is given in closed form:*

a. $\bar{f} = \mu + c + \mu\mathbb{W}[2^\gamma \exp(\frac{a-\mu-c}{\mu})]$, $\bar{p} = (1 + \bar{k})\bar{f}$;

b. $\bar{\pi}_0^s = \bar{\pi}_0^p = \bar{\pi}_k^s = \bar{\pi}_k^r = \mathbb{W}[2^\gamma \exp(\frac{a-\mu-c}{\mu})]/(2 + 2\mathbb{W}[2^\gamma \exp(\frac{a-\mu-c}{\mu})])$, $k = 1, \ldots, \bar{k}$;

c. $\bar{w}_0 = \bar{w}_k = 1/(1+\bar{k})$, $k = 1, \ldots, \bar{k}$; *and*
d. $\Delta\bar{U}_k = \Delta\bar{\eta}^w_k = \bar{f}, \bar{\eta}^u_k = 0, \quad k = 1, \ldots, \bar{k}$,
where $\mathbb{W}$ *stands for the Lambert W function.*

This proposition suggests that the pass should not offer a quantity discount to fully strategic customers who place the same value on future and present consumptions. We also observe that the individual price $\bar{f}$ is increasing in the marginal cost c, the independence parameter γ, and the scaling parameter μ. Another notable property is that all choice probabilities are identical—namely, the demand is invariant w.r.t. the form of purchase or the credit balance. However, they all increase as the pass is more differentiated from the individual purchase (as γ increases) or when the average valuation a increases, and they decrease in the marginal cost c. The identical turnpike choice probabilities also imply that the population is evenly distributed over all credit levels.

Note that when $\rho = 0$, the credit's expected marginal utility is a constant and equals the shadow price of the credit and the individual price—namely, $\Delta\bar{U}_k = \Delta\bar{\eta}^w_k = \bar{f}$. That is, a credit has the same expected marginal value to both the customers and the firm. The firm should charge the same price for a credit and an individual item. At the turnpike, fully strategic customers value future consumption the same as the present, and consequently, a credit is perceived the same as an individual item. Passholders essentially prepay for their future consumption at the same price, *shifting* the firm's future profit to the present without changing the total turnpike profit. Therefore, when $\rho = 0$, a marginal change of utilities at optimality has no impact on the turnpike profit rate, as reflected by $\bar{\eta}^u_k = 0$.

When customers are myopic ($\rho \to \infty$), the marginal utilities vanish ($\Delta U_{kt} \to 0$). In the following proposition, we show that the firm offers a quantity discount to myopic customers.

Proposition 5 (Myopic Customers). *For myopic customers ($\rho \to \infty$), the solution to the turnpike equation is also unique:*
a. *When* $\gamma = 1$, *we have* $\bar{p} < (1+\bar{k})\bar{f}$ *and* $\Delta\bar{\eta}^w_k = \mu[\exp(-\bar{p}/\mu) + \exp((a-\bar{p})/\mu) + \exp((a-\bar{f})/\mu)) + c < \bar{f}$. *The prices satisfy* $\bar{f} = \mu(1 + \exp[(a-\bar{p})/\mu] + \exp[(a-\bar{f})/\mu]) + c$ *and* $\bar{p} = (1+\bar{k})\bar{f} - \bar{k}\mu[1-\exp(-\bar{p}/\mu)]$.
b. *When* $\gamma \to 0$ *and* $\bar{k} = 1$, *we have* $\bar{f} = c + \mu + \mu\mathbb{W}(\exp[(a-c-\mu)/\mu])$ *and* $\bar{p} = 2c + \mu + \mu[2+\exp(-a/\mu)]\mathbb{W}(\exp[(a-c-\mu)/\mu]) < 2\bar{f}$, *where* $\mathbb{W}$ *is the Lambert W function.*

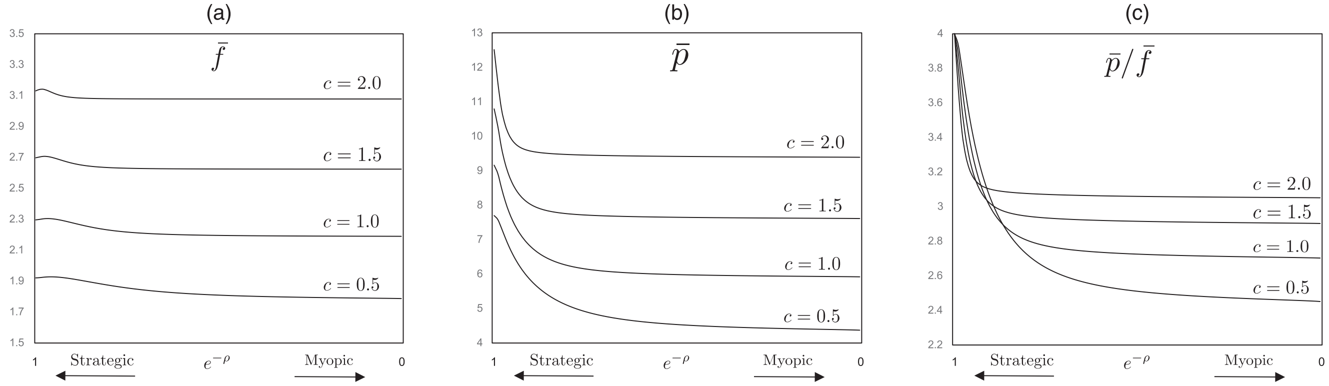
In the dynamic logit model ($\gamma = 1$), the pass offers a total discount of $\bar{k}\mu[1-\exp(-\bar{p}/\mu)]$ compared with buying the same number of individual items. Another expression of the pass price is $\bar{p} = \bar{f} + \bar{k}\{\bar{f} - \mu[1-\exp(-\bar{p}/\mu)]\}$, which suggests a *two-part tariff*: a lump-sum fee $\bar{f}$ and a constant per-unit charge $\bar{f} - \mu[1-\exp(-\bar{p}/\mu)]$ that is lower than the lump-sum fee. As $\gamma \to 0$, the turnpike prices have a closed-form solution, and we observe that the quantity discount still holds.

How does the forward-looking behavior affect the optimal pricing policy? We begin with tractable special cases by comparing the pricing policies for fully strategic and myopic customers.

Proposition 6 (Comparison in Price and Demand). *For customers with the utility discount rate ρ, let $\bar{f}_\rho$ denote the turnpike individual price, $\bar{\pi}^s_\rho$ the individual purchase probability of nonpassholder customers, and $\bar{\pi}^s_{k,\rho}$ the individual purchase probability of passholders with k remaining credits. Furthermore, let $\bar{p}_\rho$ denote the turnpike pass price, $\bar{\pi}^p_\rho$ the pass purchase probability of nonpassholder customers, and $\bar{\pi}^r_{k,\rho}$ the redemption probability of passholders with k remaining credits. The following results hold for $\gamma = 1$:*
a. $\bar{f}_0 > \bar{f}_\infty$ *and* $\bar{p}_0 > \bar{p}_\infty$, *and*
b. $\bar{\pi}^s_0 < \bar{\pi}^s_\infty$, $\bar{\pi}^p_0 > \bar{\pi}^p_\infty$, $\bar{\pi}^r_{k,0} < \bar{\pi}^r_{k,\infty}$, *and* $\bar{\pi}^s_{k,0} > \bar{\pi}^s_{k,\infty}$ *for* $k = 1, \ldots, \bar{k}$.

This proposition implies that compared with fully strategic customers, myopic customers enjoy lower prices in both individual items and passes. In fact, this finding is not only restricted to $\gamma = 1$. Figure 5 shows how the turnpike prices change in response to ρ under different marginal cost levels, where $\gamma = 0.8$. Other parameters are given by $(\bar{k}, \lambda, a, \mu) = (3, 1, 0.5, 1)$. The general trend in Figure 5, (a) and (b), is that both the

Figure 5. Turnpike Prices and Price Ratio vs. Discount Factor for Different Marginal Costs



individual and pass prices are decreasing and then stabilize as customers become more myopic (i.e., as ρ increases). However, when ρ is small enough and capacity is large enough, there exist some *reversal regimes* where the individual price is increasing in ρ (i.e., the firm may lower the individual price as customers are approaching fully strategic). We also observe that both prices increase as the marginal cost becomes higher. Figure 5(c) illustrates the general presence of a quantity discount, which becomes more prominent as customers become less strategic or the marginal cost decreases. The degree of a quantity discount also gradually stabilizes when customers become more myopic.

Part (b) of Proposition 6 compares the choice probabilities. Compared with fully strategic customers, myopic nonpassholder customers are more likely to purchase individual items and less likely to purchase the pass. By contrast, myopic passholders are more likely to redeem the credit and less likely to make individual purchases.

6. Capitalization on Strategic Behavior

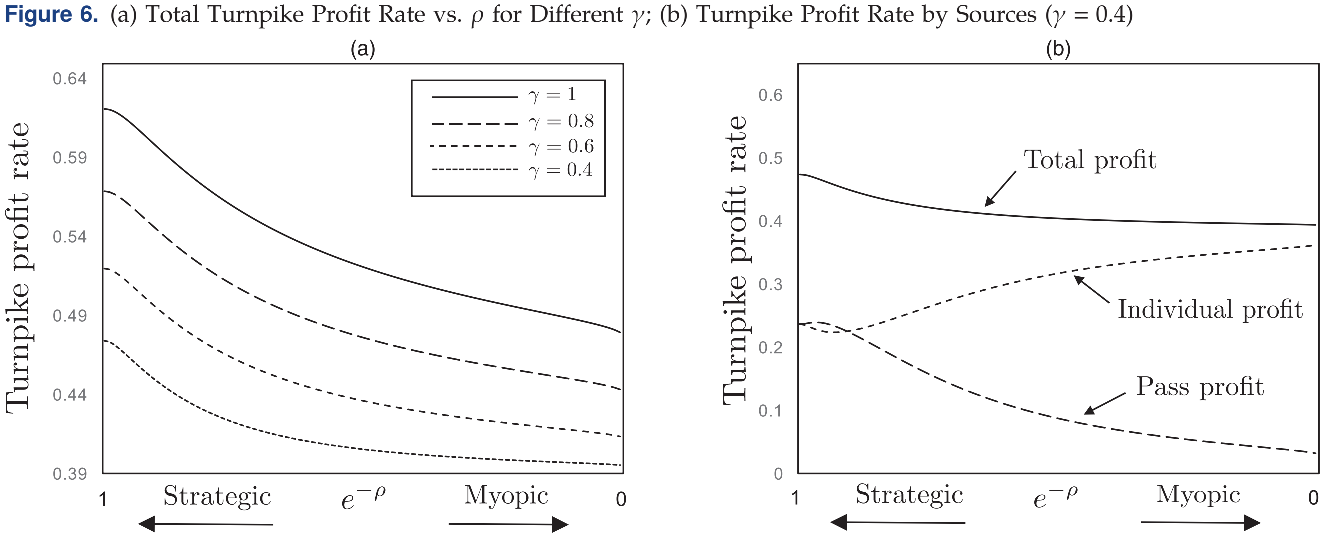
Forward-looking behavior is known to hurt profit in many settings (Aviv et al. 2009). It is also known that the firm can use rationing strategies to induce early purchase (Su 2007, Liu and van Ryzin 2008) and subsequently capitalize on the forward-looking behavior. Capitalization on strategic behavior also appears in group buying (Marinesi et al. 2017) and loyalty programs (Chun and Ovchinnikov 2018). We find that the pass, which is a form of advance purchase, is another mechanism of capitalizing on forward-looking behavior.

We first investigate the capitalization effect in the turnpike. Note that the forward-looking behavior drives customers' expected utilities, which can influence the profit through choice probabilities and then through the passholders' population. Section EC.17 in the online appendix presents a qualitative analysis on how the marginal expected utility $\Delta\bar{U}_k$ influences the choice probabilities. It suggests that the marginal utilities can have a counteracting impact on the demand depending on whether the customers have passes. Additionally, the choice probabilities alter the steady-state population distribution, which may also affect profit. Finally, the marginal utilities affect prices and add another layer of complexity. Therefore, a perturbation of the strategic behavior can cause a series of complex chain reactions affecting profit. What is the overall effect? To answer this question, we first compare the turnpike profit rate of fully strategic customers with that of myopic customers.

Proposition 7 (Comparison in Profit)**.** *Let $\bar{R}_0$ (respectively, $\bar{R}_\infty$) denote the turnpike profit rate for fully strategic (respectively, myopic) customers. When $\gamma = 1$, we have $\bar{R}_0 - \bar{R}_\infty = \lambda(\bar{f}_0 - \bar{f}_\infty) > 0$, where $\bar{f}_0$ and $\bar{f}_\infty$ are the turnpike individual prices for fully strategic and myopic customers, respectively. Furthermore, $\bar{R}_0 - \bar{R}_\infty$ is increasing in $\bar{k}$.*

Proposition 7 suggests that under the dynamic logit choice model ($\gamma = 1$), the profit capitalized from strategic behavior $\bar{R}_0 - \bar{R}_\infty$ is equal to the difference between individual prices, $\bar{f}_0 - \bar{f}_\infty$, modulated by the decision opportunity rate λ. As such, fully strategic customers must generate a higher profit rate than myopic customers because they pay a higher individual price, as shown in Proposition 6. We note further that both $\bar{f}_0$ and $\bar{f}_\infty$ have closed-form expressions, as given in Propositions 4 and 5, respectively. Therefore, we can also write $\bar{R}_0 - \bar{R}_\infty$ in closed form, from which we deduce that the pass offering more credits capitalizes more profit from the strategic behavior. Although Proposition 7 concerns only the special case of

Figure 6. (a) Total Turnpike Profit Rate vs. ρ for Different γ; (b) Turnpike Profit Rate by Sources ($\gamma = 0.4$)



$\gamma = 1$, the capitalization phenomenon is actually present for general γ, as shown in the following theorem.

Theorem 2 (Capitalization on Strategic Behavior). *The profit $\bar{R}_0 \geq \bar{R}_\infty$ for $0 \leq \gamma \leq 1$ and $c = 0$.*

The capitalization phenomenon also exists when customers are partially strategic (i.e., $0 < \rho < \infty$). Figure 6(a) shows how the total turnpike profit rate changes in response to the discount rate ρ for various values of γ. Other parameters are specified as $(\bar{k}, \lambda, a, \mu, c) = (2, 1, 0.5, 1, 0.05)$. We observe capitalization in all scenarios. Note that both profit rate and the magnitude of capitalization increase in γ. Figure 6(b) shows the breakdown of profit by sources, from which we observe that as customers become more strategic, the profit from individual items generally decreases, but the profit from passes increases and compensates for the decrease in individual profit; overall, strategic behavior increases the total turnpike profit.

Passes capitalize on the forward-looking behavior based on *advance purchase*, which can substantially increase the profit by exploiting the customer's uncertainty about future valuations (Shugan and Xie 2000). Because the pass purchase time is separated from the consumption time, customers are uncertain about the valuation of the consumption when making the advance purchase; the valuation depends on the customer's future consumption state. For example, the value of a multiday ski pass may depend on the customer's health and mood, unforeseen scheduling conflicts, and weather condition. This uncertainty favors the seller. Specifically, when customers make their purchase decisions based on the expected utility of future consumption, the seller can receive substantially more profit by advance selling.

When customers are more forward looking, they explicitly account for a larger number of consumption opportunities further into the future by discounting them less. This contributes to an increased expected value of the consumption opportunities associated with the pass. Thus, more forward-looking customers have a higher overall valuation of the pass. However, their consumption state uncertainty is also higher in absolute terms, and therefore, their information disadvantage compared with the seller also becomes higher. Indeed, a customer who plans months ahead faces more uncertainty than a customer who plans days ahead. The point about information disadvantage can be seen by first considering myopic customers who consider only the value of the current consumption opportunity. Because the purchase time coincides with the consumption time, the seller is at a disadvantage because he or she has less information about customers' valuation than the customers. This information disadvantage makes it difficult to effectively extract consumers' surplus. By contrast, for future consumption opportunities, customers themselves are uncertain about their future consumption states, putting the seller in a relatively better position with less disadvantage in terms of information as the customers' planning horizon expands. Rather interestingly, Shugan and Xie (2000, p. 231) demonstrate that "customers' uncertainty allows the seller to charge a higher price and sell to more potential customers."

The capitalization is enhanced by the unobserved heterogeneity in the customer population (Shugan and Xie 2000), which creases a differentiation between the pass and individual tickets. As described in Section 3, the heterogeneity is measured by γ in the dynamic choice model. When γ increases, we observe from Figure 6(a) that the magnitude of capitalization also increases.

Although the capitalization effect is established in the turnpike, it does not necessarily persist throughout the selling horizon. Figure 7 displays the optimal revenue rate trajectories over the entire selling horizon under different levels of strategic behavior (a larger $e^{-\rho}$ represents more strategic customers). We observe that the strategic behavior improves the profit during the early and middle stages of the horizon, but it may hurt the profit near the end of the horizon. When the selling horizon becomes short, the turnpike behavior disappears, but the profit rate trajectories still exhibit the capitalization effect in most of the planning horizon. When it is closer to the end of the horizon, the consumption times are generally also closer to the purchase

Figure 7. (Color online) Optimal Profit Rate over the Entire Horizon $(\lambda, a, \mu, c, \gamma, \bar{k}) = (0.01, 2, 1, 0.1, 0.5, 2)$

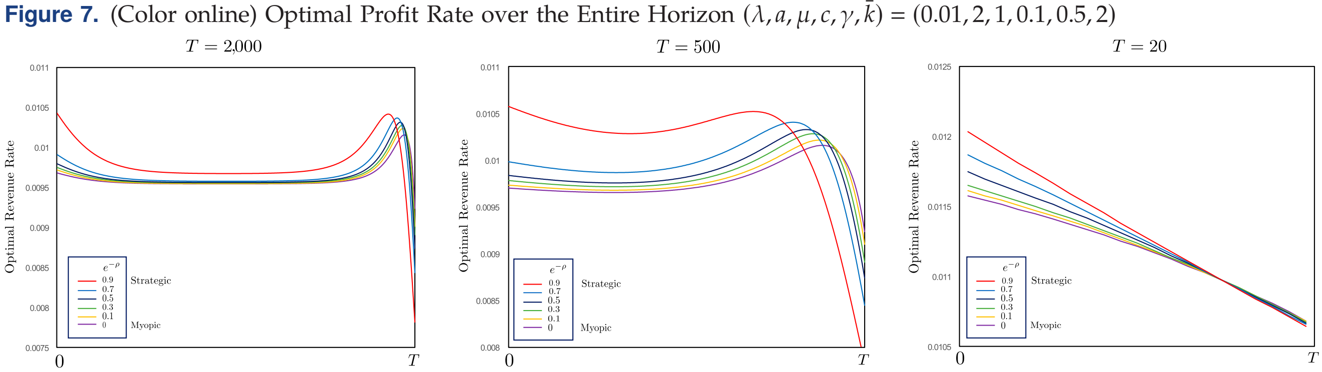


Table 2. The Optimal Expected Total Profit over the Entire Horizon (with the Same Parameters as in Figure 7)

$e^{-\rho}$	$T = 20$	$T = 50$	$T = 100$	$T = 200$	$T = 500$	$T = 1{,}000$	$T = 2{,}000$
0[a]	0.2199	0.5336	1.0330	2.0009	4.8662	9.6313	19.1606
0.1	0.2200	0.5338	1.0334	2.0014	4.8667	9.6318	19.1613
0.2	0.2201	0.5341	1.0337	2.0018	4.8671	9.6324	19.1619
0.3	0.2202	0.5343	1.0341	2.0023	4.8677	9.6330	19.1626
0.4	0.2204	0.5346	1.0346	2.0029	4.8683	9.6338	19.1636
0.5	0.2206	0.5351	1.0352	2.0036	4.8692	9.6348	19.1648
0.6	0.2208	0.5357	1.0361	2.0048	4.8706	9.6363	19.1666
0.7	0.2212	0.5366	1.0376	2.0066	4.8727	9.6387	19.1696
0.8	0.2219	0.5385	1.0404	2.0103	4.8770	9.6435	19.1755
0.9	0.2233	0.5431	1.0484	2.0208	4.8895	9.6579	19.1933
1[b]	0.2264	0.5701	1.1544	2.3625	6.0440	11.9532	23.3929

[a]Myopic.
[b]Fully strategic.

times. In this case, customers are less uncertain about their consumption states, and thus the advance purchase becomes less beneficial to the seller. Note in Figure 7 that the capitalization effect is particularly strong in the nonturnpike regime at the beginning of the horizon. This is so because most customers have not yet purchased passes at this stage, so the market size for passes is the largest. As the number of passholders increases to the turnpike level, the population of potential buyers for passes shrinks, and hence the profit rate becomes lower.

Although the capitalization effect disappears near the end of the horizon, it is interesting to observe from Table 2 that the total profit over the selling horizon is still increasing as customers become more strategic, even when the selling horizon is short. This is so because the profit gains in the early and middle stages are sufficiently large to compensate for the profit loss near the end of horizon (see Figure 7).

We have seen that passes, as a form of advance purchase, can capitalize on strategic behavior. Hence, the firm may benefit from educating customers to look forward and plan for consumption that is further in the future. This can be implemented, for example, by emphasizing the total amount of savings from passes over a long period of time, such as showing "save 40% per season" to customers.

7. Heterogeneity in the Profit Rate

In this section, we show that a passholder, while enjoying a quantity discount, can generate more profit than a nonpassholder customer in the turnpike, which results in heterogeneity in the profit rate.

7.1. Excess Profit of Passes

Our analysis here is based on the revenue-recognition principle in accrual accounting (Antle and Demski 1989). Under revenue recognition, the cash income of pass sales cannot be counted as revenue until customers redeem the credits. Consider a simple example with a two-credit pass ($\bar{k} = 1$). When a nonpassholder customer purchases the pass, he or she consumes the first credit (with cash value $\bar{f}$) immediately and keeps the second credit (with value $\bar{p} - \bar{f}$) for future redemption. The second credit revenue is not recognized until the customer actually redeems the credit. Thus, it is attributed to the *passholder*, whereas the revenue from the first credit is attributed to the nonpassholder customer.

When a decision opportunity arises, if the passholder redeems the credit (with probability $\bar{\pi}_1^r$), he or she generates recognized revenue $\bar{p} - \bar{f}$. If the passholder purchases an individual item (with probability $\bar{\pi}_1^s$), he or she generates cash revenue $\bar{f}$. Taking both together, and accounting for costs, a passholder generates an expected profit rate of $\bar{r}_1 \triangleq \bar{\pi}_1^r(\bar{p} - \bar{f} - c) + \bar{\pi}_1^s(\bar{f} - c)$ per opportunity. A nonpassholder customer buys the pass with probability $\bar{\pi}_0^p$, and only $\bar{f}$ is recognized immediately, generating a profit $\bar{f} - c$. He or she can also buy an individual item with probability $\bar{\pi}_0^s$, generating the same amount of profit $\bar{f} - c$. Therefore, the expected profit rate of a nonpassholder customer is $\bar{r}_0 \triangleq (\bar{\pi}_0^p + \bar{\pi}_0^s)(\bar{f} - c)$. If $\bar{r}_1 = \bar{r}_0$, then the pass essentially shifts the future profit of individual sales to the present, and all customers are homogeneous in the profit rate. But if $\bar{r}_1 > \bar{r}_0$, then the passholder generates an excess profit compared with the nonpassholder customer, and the profit rate is heterogeneous across customers. Next, we will show that both situations can occur.

The heterogeneity in profit is driven by the *excess profit of passes*, defined as the difference between the pass price and the total opportunity costs of the pass.

Definition 1. The *excess profit of passes* (EPP) is defined as $e_{pp} \triangleq \bar{p} - \bar{f} - \sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w$.

By selling a pass, the firm earns a cash income $\bar{p}$ but forgoes the income of an individual sale $\bar{f}$, which corresponds to the first use of the pass. In addition, it gives away $\bar{k}$ credits for future use, whose opportunity costs are measured by the marginal shadow prices $\Delta\bar{\eta}_k^w$s. To calculate EPP, one deducts all opportunity costs—namely, $\bar{f}$ and $\sum_{k=1}^{\bar{k}} \Delta\bar{\eta}_k^w$—from the pass price $\bar{p}$. The following proposition shows that the sign of EPP fully determines the relative size of $\bar{r}_1$ and $\bar{r}_0$.

Proposition 8. *The EPP* $e_{pp} \overset{\le}{\underset{>}{=}} 0 \Leftrightarrow \bar{r}_1 \overset{\le}{\underset{>}{=}} \bar{r}_0$ *for* $\bar{k} = 1$.

When the EPP is zero, the firm essentially uses the pass to recoup all opportunity costs, and as a result, a passholder generates the same profit as a non-passholder customer in the steady state. When the EPP is positive, the pass not only recoups the opportunity costs but also creates an extra profit. When the EPP is negative, the pass cannot even recoup the opportunity costs, making a passholder generate less profit. It is obvious from Proposition 8 that EPP drives the heterogeneity in profit rate.

What determines the sign of the EPP? To this end, we conduct a combination of analytical and numerical studies for the case of $\bar{k} = 1$, which captures the essence of the problem.

Theorem 3. *For* $\bar{k} = 1$, *the following results hold:*

a. *When* $\rho = 0$ *(i.e.,* $e^{-\rho} = 1$*), we have* $e_{pp} = 0$.
b. *When* $\gamma \to 0$, *we have* $e_{pp} \to 0$.
c. *When* $\rho \to \infty$ *(i.e.,* $e^{-\rho} \to 0$*), we have the following:*
 i. $e_{pp} > 0$ *if and only if* $0 < \gamma < 1$, *and*
 ii. $e_{pp} \to 0$ *if* $\gamma = 1$ *or* $\gamma \to 0$.
d. *When* $\gamma = 1$, *we have the following:*
 i. $e_{pp} > 0$ *if and only if* $0 < \rho < \infty$ *(i.e.,* $0 < e^{-\rho} < 1$*), and*
 ii. $e_{pp} \to 0$ *if* $\rho = 0$ *(i.e.,* $e^{-\rho} = 1$*) or* $\rho \to \infty$ *(i.e.,* $e^{-\rho} \to 0$*).*

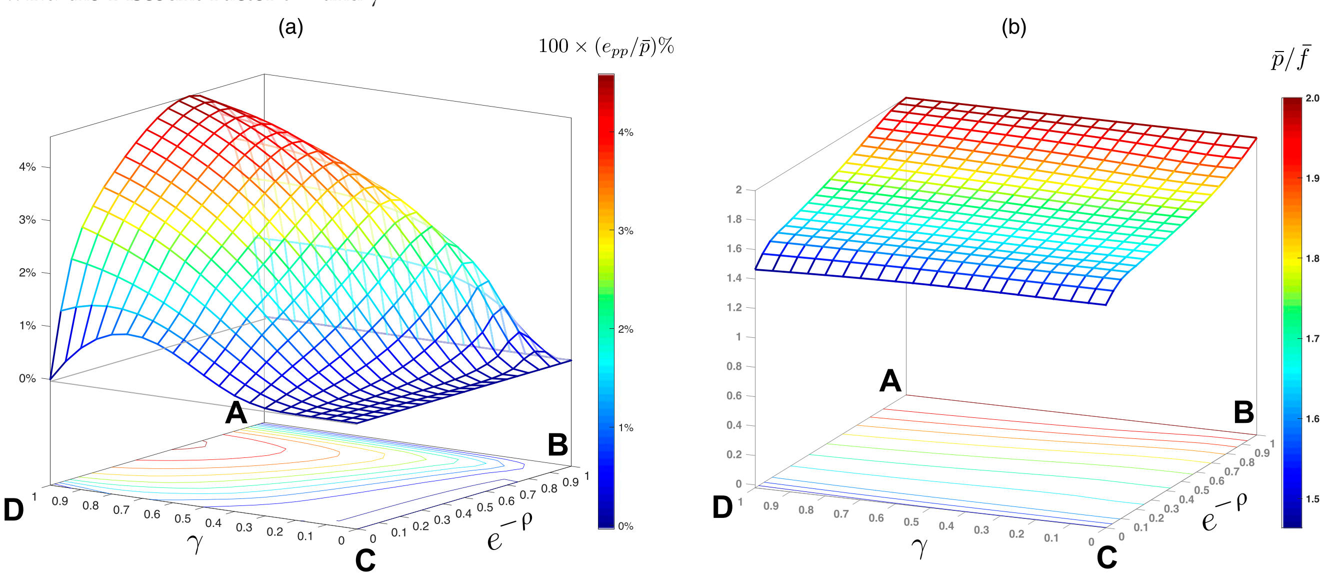
Figure 8(a) shows the relative percentage of the EPP to the pass price (i.e., $e_{pp}/\bar{p}$) for the case of $\bar{k} = 1$ under different combinations of the discount factor $e^{-\rho}$ and γ. Other parameters are specified as $(\lambda, a, \mu) = (3, 2, 1)$. We observe that the EPP is small compared with the pass price, accounting for less than 5% of $\bar{p}$. But it is *nonnegative* for all parametrization. We now examine the four edges of the rectangle shown in Figure 8(a), where each edge corresponds to a scenario listed in Theorem 3.

1. The edge AB corresponds to the case of $\rho = 0$ ($e^{-\rho} = 1$), in which customers do not discount future utility and hence are fully strategic. Part (a) of Theorem 3 suggests that the EPP is zero in this case, which is clearly shown in Figure 8(a).

2. The edge BC represents the case of $\gamma \to 0$. That is, the unobserved idiosyncrasies with regard to the choice between passes and individual items are perfectly correlated. Part (b) of Theorem 3 suggests that the EPP is also zero in this case, which is again consistent with Figure 8(a).

3. The edge CD corresponds to the case of $\rho \to \infty$ ($e^{-\rho} \to 0$), in which case customers do not value future utility and hence are myopic. The choice model also becomes static because all future utility terms become zero. For this case, part (c) of Theorem 3 suggests a

Figure 8. (Color online) (a) The Percentage of EPP in the Pass Price ($e_{pp}/\bar{p}$) and (b) the Ratio of Pass to Individual Price ($\bar{p}/\bar{f}$) w.r.t. the Discount Factor $e^{-\rho}$ and γ



positive EPP except when $\gamma \to 0$ or $\gamma = 1$, which can be easily observed from both figures.

4. The edge DA corresponds to the dynamic logit model in the econometrics literature (Rust 1994)—namely, when $\gamma = 1$. Part (d) of Theorem 3 suggests that a positive EPP is expected unless $\rho \to \infty$ (vertex D) or $\rho = 0$ (vertex A), as evident in both figures.

On the basis of Theorem 3 and the numerical examples, we have identified three situations with zero EPP: (1) $\rho = 0$ (edge AB), (2) $\gamma \to 0$ (edge BC), and (3) $\rho \to \infty, \gamma = 1$ (vertex D), from which one can identify the following three drivers that *jointly* drive the EPP.

The first driver is the discount of future utility by customers. If customers do not discount their future utility, the firm should not offer a quantity discount in passes. In this case, customers are homogeneous in terms of their behaviors, and thus the EPP does not arise. The second driver is the divergence or imperfect correlation in unobserved idiosyncrasies with regard to the choice between passes and individual items among customers, which come from interindividual and intraindividual variations in preferences. The third driver is the interdependence between the choices of passes and individual items. More specifically, a customer's preference between the no-purchase alternative and the pass depends on the attributes of individual items (i.e., the third alternative).[5] In Figure 8(a), the independence occurs only in the situation represented by the vertex D—namely, when customers are myopic ($e^{-\rho} \to 0$) and the unobserved idiosyncrasies are independent ($\gamma = 1$).[6]

Note also that the EPP does *not* conflict with a quantity discount. Compared with a nonpassholder customer, a passholder can still generate a higher profit rate while enjoying a quantity discount, as illustrated in Figure 8 for the case of $\bar{k} = 1$, in which other parameters are identical to Figure 8(a). More specifically, Figure 8(a) shows that the EPP is nonnegative for all combinations of ρ and γ, and Figure 8(b) shows the presence of a quantity discount—namely, $\bar{p}/\bar{f} < 2$—for the same parameter combinations. There is no quantity discount only when $\rho = 0$ (as shown by edge AB), in which case the EPP is also zero. In fact, it is easy to observe that the EPP is positive only when the firm offers a quantity discount. In the following proposition, we further prove the optimality of a quantity discount in some special cases.

Proposition 9 (Quantity Discount). *The following results hold for $\bar{k} = 1$:*

a. *When $\rho = 0$, we have $\bar{p} = 2\bar{f}$.*

b. *When $0 < \rho < \infty$, we have $\bar{f} < \bar{p} < 2\bar{f}$ for both $\gamma \to 0$ and $\gamma = 1$.*

7.2. Composition of Profit

So far we have been focusing on the individual profit rate. We now turn to the aggregated profit rate for the entire customer population. The following theorem shows three ways of breaking down the total turnpike profit rate.

Theorem 4 (Composition of the Profit Rate). *The turnpike profit rate $\bar{R}$ can be decomposed as follows:*

$$\bar{R} = \underbrace{\lambda\bar{w}_0\left[\bar{\pi}_0^p(\bar{p}-c) + \bar{\pi}_0^s(\bar{f}-c)\right]}_{\text{Nonpassholders' cash profit}} + \underbrace{\lambda\sum_{k=1}^{\bar{k}}\bar{w}_k\left[\bar{\pi}_k^s(\bar{f}-c) - \bar{\pi}_k^r c\right]}_{\text{Passholders' cash profit}} \tag{8}$$

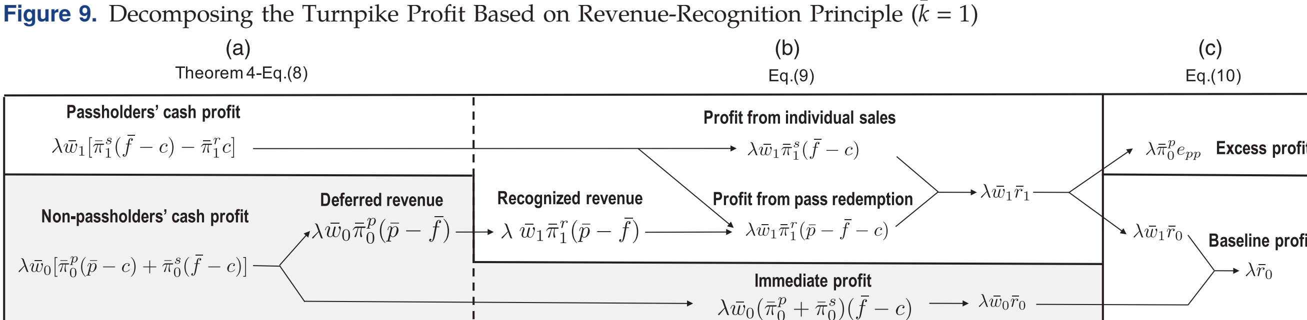
$$= \underbrace{\lambda\bar{w}_0\left(\bar{\pi}_0^p + \bar{\pi}_0^s\right)(\bar{f}-c)}_{\text{Nonpassholders' recognized profit}}$$

$$+ \underbrace{\lambda\sum_{k=1}^{\bar{k}}\bar{w}_k\left[\bar{\pi}_k^r\left(\frac{\bar{p}-\bar{f}}{\bar{k}} - c\right) + \bar{\pi}_k^s(\bar{f}-c)\right]}_{\text{Passholders' recognized profit}} \tag{9}$$

$$= \underbrace{\lambda\bar{r}_0}_{\text{Baseline profit}} + \underbrace{\lambda\bar{\pi}_0^p e_{pp}}_{\text{Excess profit}}. \tag{10}$$

We interpret this theorem using an example of two-credit passes ($\bar{k} = 1$). A straightforward decomposition of the turnpike profit rate is based on cash flows, as given in (8) and shown in the left half of Figure 9(a): the profit can be split into two parts: (1) $\lambda\bar{w}_1[\bar{\pi}_1^s(\bar{f}-c) - \bar{\pi}_1^r c]$ is the cash profit of passholders purchasing individual items and redeeming credits, and (2) $\lambda\bar{w}_0[\bar{\pi}_0^p(\bar{p}-c) + \bar{\pi}_0^s(\bar{f}-c)]$ is the cash profit of nonpassholder customers purchasing passes or

Figure 9. Decomposing the Turnpike Profit Based on Revenue-Recognition Principle ($\bar{k} = 1$)



individual items. The latter can be decomposed further into two components.

One component, $\lambda\bar{w}_0(\pi_0^p + \pi_0^s)(\bar{f} - c)$, as shown at the bottom of Figure 9(b), is the profit of immediate uses. It is related to the uses occurring immediately after the purchase, made by pass buyers who use their first credits or individual buyers who use them right after the purchases. Because the combined demand rate is $\lambda\bar{w}_0(\pi_0^p + \pi_0^s)$ and each customer brings a cash profit $\bar{f} - c$, the total profit rate is thus given by $\lambda\bar{w}_0(\pi_0^p + \pi_0^s)(\bar{f} - c)$. Recall that $\bar{r}_0 = (\pi_0^p + \pi_0^s)(\bar{f} - c)$ is the profit rate of an individual nonpassholder customer given a decision opportunity. Therefore, the total profit rate can also be written as $\lambda\bar{w}_0\bar{r}_0$.

The other component, $\lambda\bar{w}_0\bar{\pi}_0^p(\bar{p} - \bar{f})$, as shown on the right half of Figure 9(a), is associated with future uses. Here $\lambda\bar{w}_0\bar{\pi}_0^p$ is the pass demand rate, and $(\bar{p} - \bar{f})$ is the price prepaid by each customer. By the revenue-recognition principle, it is considered a deferred revenue, not recognized until customers redeem their credits. Note that by the time these customers redeem their credits, their status has changed to passholders. We can thus imagine that each passholder pays an amount $\bar{p} - \bar{f}$ when he or she redeems the credit but pays nothing when he or she purchases it. The redemption occurs at a rate $\lambda\bar{w}_1\bar{\pi}_1^r$, and each redemption recognizes $(\bar{p} - \bar{f})$, so the recognized revenue rate is $\lambda\bar{w}_1\bar{\pi}_1^r(\bar{p} - \bar{f})$, shown in the left part of Figure 9(b). Note further that in the steady state, the pass purchase and credit redemption are balanced (i.e., $\bar{w}_0\pi_0^p = \bar{w}_1\pi_1^r$); therefore, the deferred revenue rate is also equal to the recognized revenue rate.

We have seen from the preceding that under revenue recognition, the deferred revenue can be attributed to *passholders*. When a passholder redeems the credit, he or she pays $(\bar{p} - \bar{f})$, and the redemption incurs a cost c to the firm. So the profit rate of pass redemption is $\lambda\bar{w}_1\bar{\pi}_1^r(\bar{p} - \bar{f} - c)$, as shown at the center of Figure 9(b). In addition, passholders may also buy individual items, which generates an immediate profit of $\lambda\bar{w}_1\bar{\pi}_1^s(\bar{f} - c)$ per unit of time, as shown at the top of Figure 9(b). Combining the two sources, the passholder profit rate is $\lambda\bar{w}_1[\bar{\pi}_1^s(\bar{f} - c) + \bar{\pi}_1^r(\bar{p} - \bar{f} - c)]$ (see Equation (9)), which can be also written as $\lambda\bar{w}_1\bar{r}_1$, where $\bar{r}_1$ is the profit rate of an individual passholder given a decision opportunity.

Finally, we have shown in Proposition 8 that a positive EPP makes a passholder generate a higher per-capita profit rate than a nonpassholder customer counterpart in the steady state—namely, $\bar{r}_1 > \bar{r}_0$. In light of this, a nonpassholder customer's profit rate r_0 can be considered as the *baseline* profit rate of an individual with a decision opportunity. A passholder generates an excess amount of profit on top of the baseline. Equation (10) in Theorem 4 shows that the aggregated profit rate also consists of the aggregated baseline rate $\lambda(\bar{w}_0 + \bar{w}_1)\bar{r}_0 = \lambda\bar{r}_0$ (because the entire population is normalized to 1), and the aggregated excess profit $\lambda\bar{\pi}_0^p e_{pp}$, as shown in Figure 9(c).

Figure 10 shows how the relative percentage of baseline (Figure 10(a)) and excess (Figure 10(b)) profit rates vary with respect to the discount factor $e^{-\rho}$ and the parameter γ, in which other parameters are specified as $(c, \bar{k}, \lambda, a, \mu) = (0.01, 2, 1, 0.5, 1)$. We observe that the percentages of baseline and excess profits are not monotone in the level of strategic behavior. The excess profit reaches the maximum at an intermediate level of strategic behavior and tends to decrease when customers are getting closer to fully strategic or myopic. This observation echoes the findings in Theorem 3.

Figure 10. The Percentage Composition of Turnpike Profit Rate w.r.t. $e^{-\rho}$ and γ

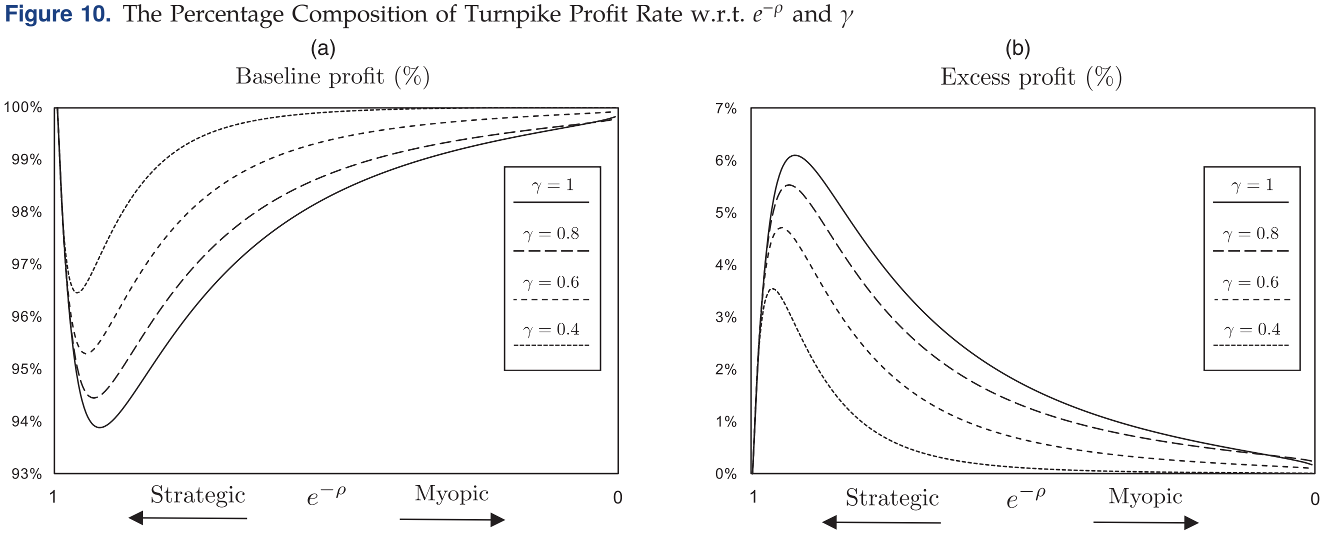
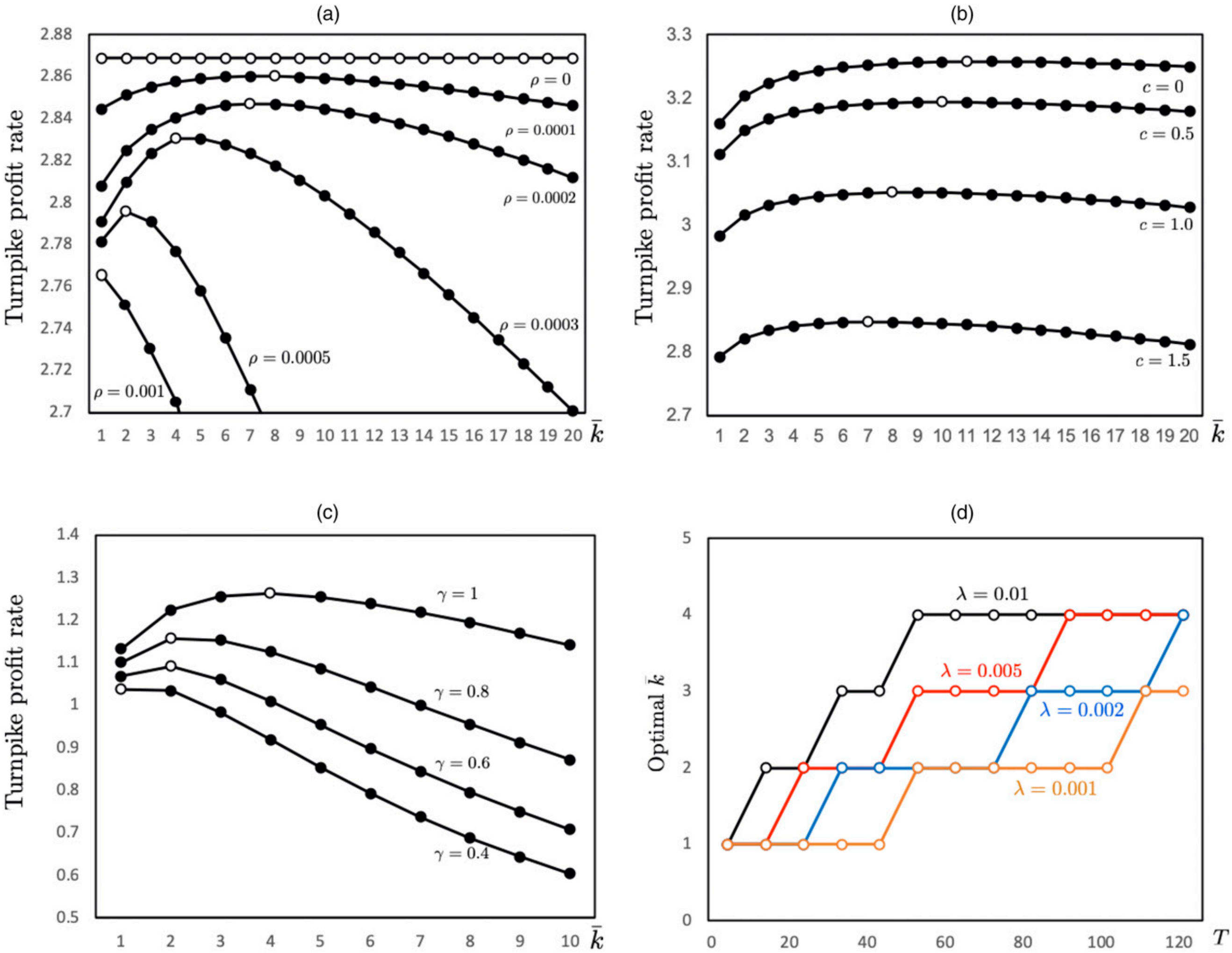


Figure 11. (Color online) Turnpike Profit Rate vs. the Number of Credits on Pass Under Different Levels of (a) Strategic Behavior, (b) Marginal Cost, and (c) Heterogeneity



Note. Panel (d) shows the optimal $\bar{k}$ versus horizon length.

8. Optimization of $\bar{k}$

We have been focusing on the optimization of prices, but the number of credits sold in the pass—namely, $\bar{k}$—is also a decision variable. Similar to prices, $\bar{k}$ can be adjusted over time. In this section, we consider the optimization of either a fixed $\bar{k}$ or the $\bar{k}$ with a finite number of changes.

8.1. When $\bar{k}$ Is Static

We first consider the special cases with fully strategic customers and myopic customers in the following proposition.

Proposition 10 (Optimal $\bar{k}$).

a. *The turnpike profit rate for fully strategic customers ($\rho = 0$) is*

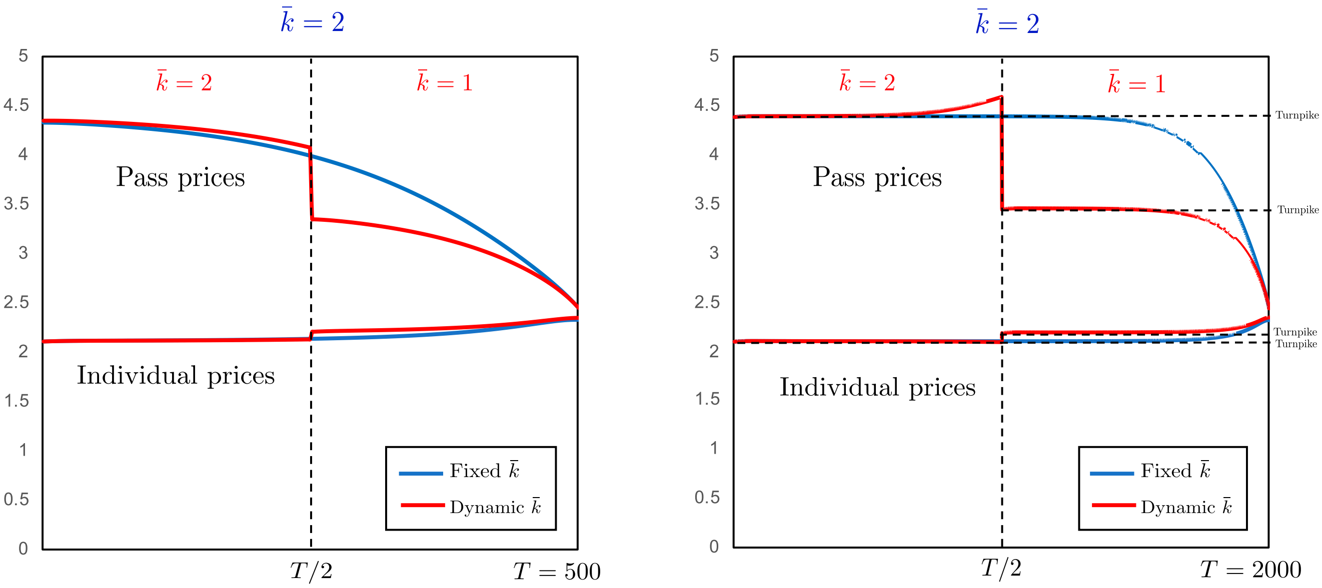
$$\bar{R}_0 = \lambda\left\{\mu + \mu\mathbb{W}\left[2^{\gamma}\exp\left(\frac{a-\mu-c}{\mu}\right)\right]\right\} \times \mathbb{W}\left[2^{\gamma}\exp\left(\frac{a-\mu-c}{\mu}\right)\right] \Big/ \left(1+\mathbb{W}\left[2^{\gamma}\exp\left(\frac{a-\mu-c}{\mu}\right)\right]\right),$$

which does not depend on $\bar{k}$. The individual price $\bar{f}$ is independent of $\bar{k}$, and the pass price $\bar{p}$ is linearly increasing in $\bar{k}$.

b. *When $\gamma = 1$ and customers are myopic ($\rho \to \infty$), it is optimal to choose $\bar{k} = 1$ in the turnpike. Furthermore, $\bar{f}$ is decreasing in $\bar{k}$.*

When customers are fully strategic ($\rho = 0$), the number of credits $\bar{k}$ does not influence the firm's profit. When customers are myopic and $\gamma = 1$, the optimal $\bar{k}$ is 1. In this case, offering multiple credits to the myopic customers is not optimal. In practice, customers are between myopic and fully strategic; numerical studies suggest that there is a unique $\bar{k}$ that maximizes the profit rate. An example is shown in Figure 11(a) with parameters specified as $(\gamma, \lambda, a, \mu, c) = (0.7, 6, 2, 1, 1.4)$. We observe that the optimal $\bar{k}$ decreases in ρ. That is, the firm should offer fewer credits in the pass when customers become less strategic. Indeed, customers who are less forward looking are less likely to buy a

Figure 12. (Color online) Optimal Price Trajectories with Fixed $\bar{k}=2$ (in Blue Online) vs. Decreasing $\bar{k}$ (in Red Online) in the Short Horizon (Left Panel) and Long Horizon (Right Panel)



Note. Other parameters are $(\lambda, a, \mu, \gamma, \rho) = (0.01, 2, 1, 0.9, 0.05)$.

large number of credits in advance. Another observation is that the profit rate becomes less sensitive to the number of credits when customers become more strategic, in which case the optimization seems less important. In the extreme case of $\rho = 0$, it is not necessary to optimize $\bar{k}$ because it has no impact on the profit rate.

Proposition 10 also suggests that the individual prices can be coupled with what is offered on the pass, which is a key feature of the joint pricing. Specifically, if more credits are offered on the pass (i.e., $\bar{k}$ increases), then the individual turnpike price should be lower for myopic customers. However, this coupling disappears when customers are fully strategic, in which case the optimal individual price $\bar{f}$ becomes independent of $\bar{k}$, and the pass price becomes linear in $\bar{k}$.

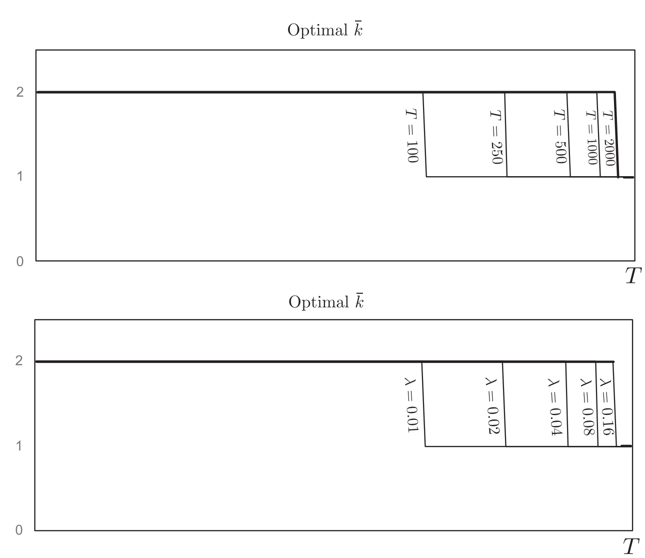
Figure 11(b) shows that both the turnpike profit rate and the optimal $\bar{k}$ decrease in the marginal cost c. That is, when the marginal cost is lower, more credits should be included in the pass. From this observation, the firm may use the case of zero marginal cost to find an upper bound for the optimal number of credits. Parameters for this example are $(\gamma, \lambda, a, \mu, \rho) = (0.7, 6, 2, 1, 0.0001)$. In Figure 11(c), we observe that the optimal $\bar{k}$ is increasing in γ. That is, the pass should offer more credits when the customer preferences become more heterogeneous. Parameters are $(\lambda, a, \mu, \rho, c) = (1, 2, 1, 0.001, 0.1)$.

Figure 11(d) shows that the optimal $\bar{k}$ is increasing in the horizon length T, in which we maximize the total profit over the selling horizon, with parameters $(\gamma, a, \mu, \rho, c) = (1, 2, 1, 0.01, 1)$. The optimal $\bar{k}$ converges to the turnpike optimal setting when T is sufficiently large. The convergence is generally slower for lower arrival rate λ.

8.2. When $\bar{k}$ Changes a Finite Number of Times

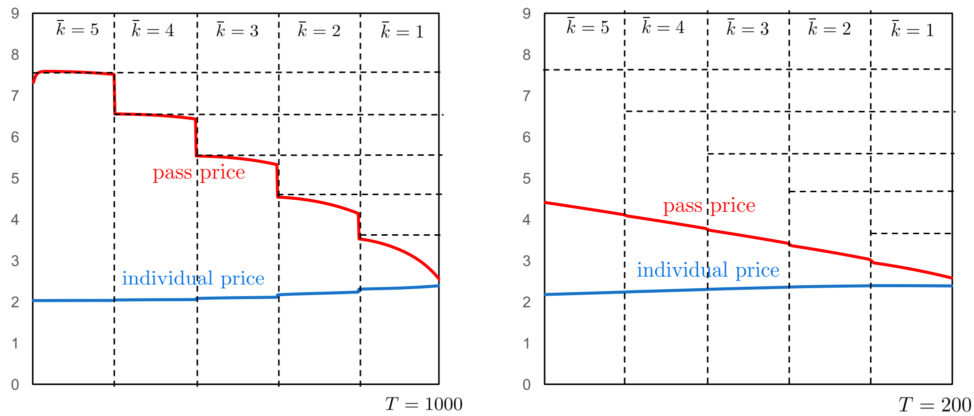
The seller can dynamically set $\bar{k}$ in conjunction with prices over time. Computing the corresponding optimal policy is challenging because one has to jointly optimize the value of $\bar{k}$ together with the times at which it changes. Nevertheless, insights can be gained

Figure 13. Optimal Change Time for $\bar{k}$ Corresponding to Different Horizon Lengths T (Upper Panel) and the Maximum Arrival Rates λ (Lower Panel)



Note. The parameters are $(a, \mu, \gamma, \rho) = (1.5, 1, 1, 0.001)$.

Figure 14. (Color online) Optimal Price Trajectories When $\bar{k}$ Undergoes Multiple Changes



Note. The horizontal dashed lines represent turnpike pass prices for different $\bar{k}$, and the vertical dashed lines represent the change times.

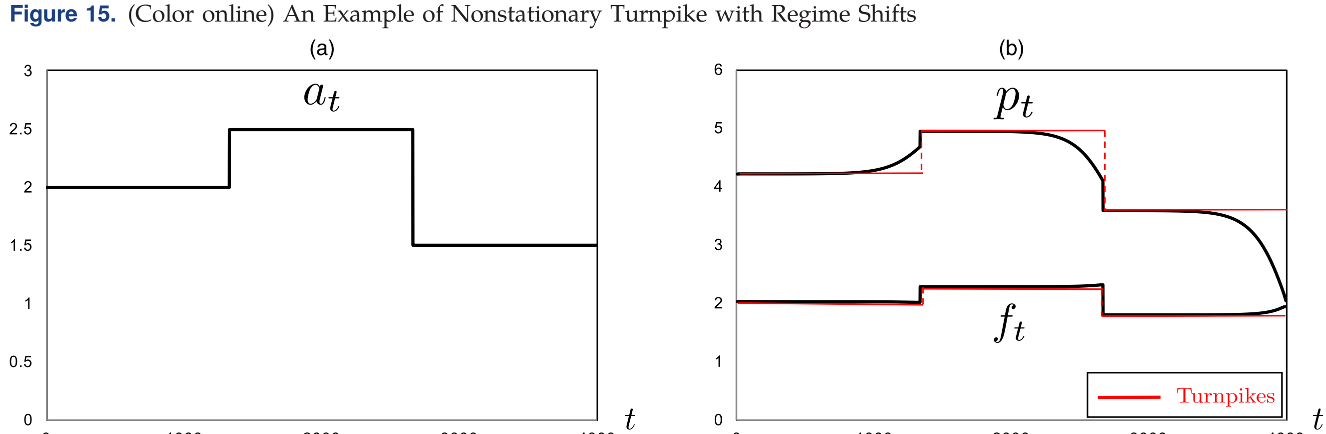
from the simple cases where $\bar{k}$ is allowed to change once. We describe the corresponding model in Section EC.27 of the online appendix.

In Figure 12, we compare the optimal price trajectories under a fixed $\bar{k}$ with those under a decreasing $\bar{k}$. The change in $\bar{k}$ causes discontinuities in the price trajectories; the pass price jumps downward significantly, whereas the individual price jumps slightly upward. When the horizon is sufficiently long (see the right panel of Figure 12), the optimal prices exhibit piecewise turnpike property: they stay close to the steady states except near the change time and the end of the horizon.

The change time can also be optimized. A numerical example is presented in Figure 13, in which we enumerate through different combinations of change time and $\bar{k}$s (before and after the change) to maximize the total profit over the horizon T. Each vertical line in the upper panel represents the relative location of the optimal change time in the selling horizon with a specific length T. As the horizon becomes longer, we observe that the optimal change time gets closer to the end of the horizon. This example suggests that the turnpike property may still emerge when prices and $\bar{k}$ are both fully dynamic. When the horizon is long enough, the optimal $\bar{k}$ stays at the corresponding optimal turnpike value (which is 2 in this example) for most of the time. Note that the arrival rate λ also plays an important role, as shown in the lower panel of Figure 13. When λ is smaller, the optimal change in $\bar{k}$ happens earlier. For a ski resort where customers ski once a month on average, it may be optimal to reduce $\bar{k}$ in the middle of winter. But for an outdoor swimming pool where customers visit five times a month, it is probably optimal to fix $\bar{k}$ in most of the summer and only decrease it near the end of the summer.

When $\bar{k}$ changes multiple times, as shown in Figure 14, the optimal price trajectories may exhibit turnpike behavior when the changes are far apart, except near the end of horizon (see the left panel of Figure 14). Using the turnpike price during the corresponding time window may provide a reasonable approximation to the optimal price trajectories. However, the turnpike behavior disappears when the selling horizon is short (see the right panel of Figure 14), in which case the optimal prices can be continuously changing and lie below the turnpike prices because the horizon is too short to allow full utilization of the pass.

Figure 15. (Color online) An Example of Nonstationary Turnpike with Regime Shifts



9. Summary and Discussion

We consider joint pricing of passes and individual items in the face of rational customers who strategize on not only purchases but also subsequent redemption and renewals. Although the problem appears rather complex, the optimal pricing policy has a surprisingly simple steady-state component, which lends itself to formal mathematical analysis.

We showed that passes may encourage consumption by offering a quantity discount, as long as customers are not fully strategic. They also allow the firm to benefit from strategic behavior by exploiting the customer's uncertainty about future consumption. Under the revenue-recognition principle, the revenue of advance selling is attributed to passholders, who can generate a higher per-capita revenue rate than nonpassholder customers. This excess profit is closely related to the level of the strategic behavior, divergence in customer idiosyncrasies, and the interdependence between the choices of passes and individual items.

We have assumed a stationary environment in this paper. For example, the average valuation of each consumption is assumed to be fixed, which helps to establish the turnpike property. In reality, however, the average valuation is subject to change. Figure 15(a) illustrates a scenario with a nonstationary valuation, where a_t has two regime shifts in the course of sales. The corresponding optimal price trajectories are shown in Figure 15(b), which exhibit a piecewise turnpike property. It is reasonable to speculate that when the regime shifts are not frequent, we can still expect the steady state occupying a significant portion of the sales horizon, where most conclusions drawn from the turnpike equations in this paper still hold.

Important directions of further research include using passes as a competitive tool, pricing passes with rolling expirations, and large-scale network pass pricing problems. The modeling framework and findings of this paper can also motivate and inform further studies that involve the advance purchase of multiunit products. Because the dynamic choice model for the strategic behavior is deeply rooted in econometrics and admits maximum-likelihood estimation, it can be feasible to conduct empirical testing and parameter estimation based on our model. Finally, this paper has only examined a setting where the seller can credibly commit to a price sequence. In general, this may not always be true in practice—in particular, when the effects of individual customer purchases have a significant impact on the state of the market. Yet it is possible that a state-invariant pricing policy may be asymptotically optimal. It is an interesting research question to identify conditions the market must satisfy for this to hold.

Acknowledgments

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Endnotes

[1] A straightforward mathematical consequence of this theorem is the uniqueness of the solution to the initial value problem for the population. Indeed, because the solution to the utility equations (2) is unique, the choice probabilities are also uniquely determined, and the population equations can be viewed as nonstationary linear differential equations.

[2] It is easy to see that U_{kt} depends on f_t because f_t enters the equation for U_{kt} directly. Note that the pass price p_t drives the dynamics of U_{0t}, which, in turn, drives $U_{1t}, U_{2t}, \ldots, U_{(k-1)t}$ in sequence. When $U_{(k-1)t}$ enters the equation for U_{kt}, it also carries the influence of pass price p_t.

[3] In static multinomial logit (MNL) models, the probability ratio of a pass over an individual item depends only on the attributes of these two alternatives. The dynamic logit model, however, does not have this property because the probability ratio π_{0t}^p/π_{0t}^s depends on $f_t - p_t + U_{0t} - U_{\bar{k}t}$, in which both U_{0t} and $U_{\bar{k}t}$ depend on the attributes of other alternatives.

[4] Our problem can be viewed as a Stackelberg game where the firm leads by posting the price paths and customers respond to the price paths as the followers. Each customer is free to rationally adjust choices at each point in time. Nevertheless, because an individual customer is infinitesimal, he or she cannot affect the state of the game. Furthermore, customers do not coordinate their actions, and as a result, the trajectory of the system remains effectively determined by the firm's pricing policy. Stated formally, this is reflected by the uniqueness of solutions to the differential equations governing the trajectory of the system for a given pricing policy (see Theorem 1). From the firm's point of view, the problem is that of the optimal control of the system governed by the ordinary differential equations. Such control problems obey the principle of optimality, and the resulting optimal pricing policy is time consistent.

[5] If the pass and individual ticket are two independent alternatives, then the pass price can be optimized as an independent new product after adjusting for the opportunity cost. By symmetry, the optimality equations for both alternatives become essentially identical. More specifically, imagine a new product whose price is the pass price net the shadow prices of credits $\bar{p} - \sum \Delta\bar{\eta}_k^w$. It must satisfy the optimality condition $\bar{p} - \sum \Delta\bar{\eta}_k^w = c + \mu + \alpha$ in (EC.53) of the online appendix. Note that the optimality condition for the individual price is $\bar{f} = c + \mu + \alpha$ in (EC.55) of the online appendix, which has an identical right-hand side. Therefore, $\bar{p} - \sum \Delta\bar{\eta}_k^w = \bar{f}$ at optimality, implying no EPP. As a result, the passholder's turnpike profit becomes identical to the nonpassholder customer's.

[6] In this special case, the choice is represented by a multinomial logit model with the IIA property. But for strategic customers (i.e., $\rho < \infty$), the pass and individual alternatives are *dependent* even when $\gamma = 1$ (Rust 1994).

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