Expected Profit of Fixed Price Policy Decays Exponentially in the Lead Time

Stefanus Jasin, Jiaxin (Alys) Liang
Stephen M. Ross School of Business, University of Michigan, Ann Arbor, MI, USA
sjasin, jiaxinl@umich.edu

Yanzhe (Murray) Lei
Smith School of Business, Queen's University, Kingston, Ontario
yl64@queensu.ca

Lai Wei
Carroll School of Management, Boston College, Massachusetts, US
weilx@bc.edu

We revisit the joint inventory and pricing problem with backlogging and positive lead time, and ask, "What is the value of dynamic pricing?" While it is intuitive that dynamic pricing can yield a better expected profit than the best Fixed Price (FP) policy because the former is more flexible than the latter, the magnitude of this improvement is not well understood. In this paper, we shed partial light on this question by focusing on the impact of delivery lead time in a setting with a linear purchase rate and a Normal-like demand. We analytically derive an upper bound for the expected profit under the best FP policy (i.e., the policy that applies the optimal sequence of fixed prices, which may not be stationary over time, in combination with the best adaptive replenishment policy) and also a lower bound for the expected profit under the optimal joint inventory and pricing policy. Under a mild assumption, we show that the expected profit under the best FP policy decays *exponentially* in the length of lead time and, for all sufficiently large lead times, grows at most *logarithmically* in the length of the selling horizon, whereas the expected profit under the optimal joint inventory and pricing policy grows linearly in the length of the selling horizon regardless of the length of lead time. These results hold even in the setting where demand variability is smaller than its mean, which is in contrast to the well-known result in the Revenue Management (RM) literature regarding the near-optimality of the best FP policy in such a setting. More precisely, whereas in the aforementioned classic RM setting, dynamic pricing has been shown to only capture the second-order magnitude of the optimal revenue, in the presence of lead time and holding cost, dynamic pricing is sometimes needed even to capture the first-order magnitude of the optimal profit. This highlights the necessity of dynamic pricing in such a setting.

1. Introduction

It is known in the Revenue Management (RM) literature that when demand variability is (asymptotically) smaller than its mean and demands across different periods are independent given prices, the best Fixed Price (FP) policy under which the prices applied at all periods are simultaneously decided at the beginning of the horizon is asymptotically optimal as mean demand grows large (e.g., Gallego and Van Ryzin 1994, 1997). In other words, in such settings, the best FP policy already captures the first-order magnitude of the optimal revenue, and dynamically adjusting the prices over time only contributes to capturing the second-order magnitude of the optimal revenue (e.g., Jasin 2014, Chen et al. 2015, Lei et al. 2018, Balseiro et al. 2023).

In this paper, we revisit the classic joint inventory and pricing problem with backlogging and positive lead time and ask whether the same insight also holds for this setting. Note that, in contrast to the asymptotic analysis in the classic RM setting in which the performance of the best FP policy is largely determined by the magnitude of mean demand in each period, in the joint inventory and pricing setting, both the mean demand *and* lead time can be "large". Thus, it is important to understand the performance of FP policy as a function of not only mean demand but also lead time, which is the focus of this paper. More concretely, let m denote the mean demand in each period, and let L and T denote the length of lead time and the length of selling horizon, respectively. We are interested in understanding the performance of the best FP policy in the setting where all three parameters m, L, and T can simultaneously grow large. We derive an upper bound for the expected profit under the best FP policy and also a lower bound for the optimal expected profit. Our bounds highlight the non-trivial impact of lead time in the joint inventory and pricing setting, which underscores the need for dynamic pricing.

1.1. Our results and contributions

For tractability, in this paper, we only focus our analysis on the setting with a linear purchase rate. We also further assume that the randomness in demand is captured by a symmetric truncated Normal random variable to allow us to derive an explicit characterization of the bounds. Under a mild assumption, our main results are twofold:

1. We show that the expected profit under the best FP policy (i.e., the policy that applies the optimal sequence of fixed prices in combination with the best adaptive replenishment policy) decays *exponentially* in the length of lead time. Moreover, for all sufficiently large lead times, the expected profit under the best FP policy grows at most *logarithmically* in the length of the selling horizon. This is in contrast to the results in the RM setting in which the expected revenue under the best FP policy is linear in the length of the selling horizon.
2. To better highlight the value of dynamic pricing, we also show that the expected profit under the optimal joint inventory and pricing policy is *linear* in the length of the selling horizon *regardless* of the length of lead time. We do this by analyzing a simple policy that fixes the order quantities for all periods at the beginning of the horizon but dynamically adjusts the price over time. We prove that the expected profit under this policy is linear in the length of the selling horizon. We do *not* claim that the proposed policy is near-optimal. Its construction merely serves the purpose of showing that the expected profit under the optimal joint inventory and pricing policy is *linear* in the length of the selling horizon.

To the best of our knowledge, our work is the first in the literature that explicitly characterizes the benefit of dynamic pricing as a function of both mean demand and lead time. Indeed, as will be discussed shortly below, although several existing papers in the joint inventory and pricing literature have highlighted the value of dynamic pricing to a certain extent, they do not provide a rigorous analysis by analyzing what can be achieved by the best FP policy, which is the key contribution of our paper. In addition to providing theoretical bounds, our results also provide an important insight that, in the presence of holding cost, the performance of the best FP policy can be very sensitive to the length of lead time.

From the technical perspective, deriving an upper bound for the expected profit under the best FP policy in the aforementioned joint inventory and pricing setting turns out to be not trivial since, for any given sequence of fixed prices, which may not be stationary, we need to take into account the impact of the best replenishment policy on expected profit. Although the structure of the optimal replenishment policy for a given FP policy is known (i.e., it is a base-stock policy; see, e.g., Karlin 1958), directly utilizing this structure in our analysis is not easy. Thus, in this paper, we take a different route and develop a series of approximations to derive an upper bound.

1.2. Brief literature review

Our work in this paper is related to both the RM literature and the joint inventory and pricing literature. We discuss them in turn below.

One well-known result in the RM literature concerns the asymptotic optimality of the best FP policy in the setting where demand variability is (asymptotically) smaller than its mean; this type of result is usually proved in the context of Poisson or Bernoulli demand (see, e.g., Gallego and Van Ryzin 1994, 1997, Jasin 2014). In the last two decades, this insight has served as a cornerstone of much research in the RM literature. For example, Jasin (2014), Chen et al. (2015), Lei et al. (2018), Lei and Jasin (2020), and Chen et al. (2023a) developed dynamic pricing policies that use FP policy as the baseline control. Also, Besbes and Zeevi (2009, 2012), Keskin and Zeevi (2014), Besbes and Zeevi (2015), Chen et al. (2019), and Chen et al. (2021a) developed joint learning and pricing policies with the goal of learning the best FP policy. This list is far from exhaustive, and we refer interested readers to Den Boer (2015) for a recent overview of the field. In a nutshell, the fact that a simple FP policy is asymptotically optimal has made many RM and pricing-related problems analytically tractable. Not only FP policy can be used as a baseline for developing more sophisticated policies, it is sometimes sufficient to simply focus on optimizing and/or learning the best FP policy, which is practically convenient, especially for large-scale applications.

As for the joint inventory and pricing literature, much of the focus of this literature has been on trying to characterize the structure of the optimal policies. However, with the exception of the setting with zero lead time, the optimal policies for various joint inventory and pricing problems are still largely unknown (e.g., see Chen et al. 2021b for an overview). Recently, there have been some attempts to develop heuristic policies for joint inventory and pricing problems with positive lead time. For example, Bernstein et al. (2016) considered a backlog model and developed a simple policy that combines a base-stock inventory policy with a myopic pricing policy. Chen et al. (2023b) considered a backlog model with a positive lead time and showed that a constant-order policy, in combination with a dynamic pricing policy, is asymptotically optimal in the regime of a long lead time. Liang et al. (2020) considered a lost sales model with a Binomial/Poisson demand in each period and the possibility that each purchase might be returned in a future period, and developed a dynamic pricing policy that can be combined with a simple fixed replenishment policy. They showed that the policy is asymptotically optimal when the mean demand is large, with a loss bound that is smaller than the standard square-root loss and is independent of lead time.

While the above papers highlighted the value of dynamic pricing to a certain extent (e.g., Bernstein et al. (2016) numerically showed that the value of dynamic pricing increases with lead time whereas the policies developed in Chen et al. (2023b) and Liang et al. (2020) do not utilize any form of adaptive replenishment policy at all and only rely on dynamic pricing to balance supply and demand over time, which suggests its power), none provided a rigorous analysis on the *fundamental* value of dynamic pricing by analyzing what can be achieved by the best FP policy. As we show in this paper, the analysis of the best FP policy turns out to be non-trivial. To the best of our knowledge, the only paper in the joint inventory and pricing literature that analytically derived a bound for the performance of the best FP policy is Lei et al. (2022). However, they considered a model with zero lead time and the complexity in their model comes from the inventory allocation decisions in the one-warehouse multi-store setting. Our work complements Lei et al. (2022) by analyzing the performance of the best FP policy in the presence of positive lead time.

1.3. Organization of the paper

The remainder of this paper is organized as follows. In Section 2, we discuss the model. In Section 3, we present our main results and discuss their insights. Section 4 is dedicated to the analysis of a lower bound for the optimal profit and Section 5 is dedicated to the analysis of the best FP policy. Finally, in Section 6, we conclude the paper. Unless otherwise noted, all proofs and the remaining details of the analysis can be found in the Online Appendix.

2. Model

We consider a backlog inventory model with a positive lead time $L \geq 1$ and a finite selling horizon of T periods (we assume $L+1 \leq T$). Let I_t denote the starting inventory level at the beginning of period t, before the scheduled replenishment arrives, and let X_t denote the order quantity placed in period t, which will arrive L periods later at the beginning of period $t+L$. For convenience, we will also use $Q_t = X_{t-L}$ to directly denote the order quantity that arrives in period t. Upon observing I_t and Q_t, at the beginning of period t, the firm decides the new order quantity X_t and the price p_t to be applied during the period. We assume for simplicity that $I_1 = 0$; moreover, all orders that are scheduled to arrive in periods 1 to L are placed before the horizon begins (hypothetically, at the beginning of period 0).

2.1. Demand model

We assume a stylized continuous multiplicative demand model:

$$D_t(p_t, m, \theta) = M_t(m, \theta)\lambda(p_t),$$

where $\lambda(p_t) \in [0,1]$ is the purchase rate under price p_t and $M_t(m,\theta)$ is a random variable denoting the random market size in period t. (The multiplicative demand model is widely used in the inventory literature; see, e.g., the discussions in Chen et al. 2021b.) For analytical tractability, we assume that $\lambda(p_t)$ is linear, specifically, $\lambda(p_t) = 1 - p_t$. We also assume that $M_t(m,\theta)$'s are i.i.d random variables, which are distributed as the projection of i.i.d Normal random variables with mean m and standard deviation m^θ on the interval $[0, 2m]$. That is, we have

$$M_t(m,\theta) =^d \mathrm{Proj}_{[0,2m]}[N_t(m, m^\theta)].$$

We truncate the Normal random variable $N_t(m, m^\theta)$ on the left at 0 because market size cannot be negative. As for the truncation on the right at $2m$, it is for technical convenience, so $M_t(m,\theta)$ is a symmetric random variable and we can interpret m as the mean market size. (Note, however, whether we truncate the Normal random variable on the right or not, will not change the insights of the paper. Indeed, as shown in Section 5, as long as m is sufficiently large, we can essentially ignore the projection operator in the definition of $M_t(m,\theta)$ when deriving the bounds for FP policy.) The parameter θ captures the magnitude of market variability. We make the following assumption:

Assumption 1. *$\theta \in (-\infty, 1)$.*

That is, θ can be any number that is *strictly* smaller than 1. This assumption is for analytical tractability; the setting $\theta > 1$ is not easy to analyze and might require different technical arguments. Note that when θ is small, demand is less variable and, when θ is close to 1, demand becomes more variable. However, in all cases, as $\theta < 1$, demand variability is smaller than its mean.

2.2. The optimization problem

The firm's objective is to solve the following dynamic joint inventory and pricing problem:

$$v^* := \max_{\pi\in\Pi} \mathbf{E}\left[\sum_{t=1}^{T} p_t^\pi \cdot D_t(p_t^\pi, m, \theta) - \sum_{t=1}^{T} G\left(I_t^\pi + X_{t-L}^\pi - D_t(p_t^\pi, m, \theta)\right)\right] \tag{1}$$
$$\text{s.t.} \quad I_{t+1}^\pi = I_t^\pi + X_{t-L}^\pi - D_t(p_t^\pi, m, \theta) \quad \forall t,$$
$$I_1^\pi = 0,\, X_t^\pi \geq 0,\, p_t^\pi \in [0,1] \quad \forall t,$$

where $G(x) = h\cdot(x)^+ + b\cdot(-x)^+$ is the usual Newsvendor cost function, $h > 0$ is the per period per unit holding cost, $b > 0$ is the per period per unit backlog cost (we assume $b \geq h$), and Π is the set of all feasible non-anticipating policies. We use the superscript π as a reference to policy π. However, whenever not needed, we will often drop this superscript from all notations for simplicity.

In the above model, we have assumed that the unit ordering cost is zero. This is for analytical convenience and is sufficient for our purpose in this paper. The presence of a (linear) ordering cost is not difficult to handle and will not change the insights of our analysis. (Indeed, it is common in the literature to assume zero ordering cost, e.g., see the discussions in Chen et al. 2021b.)

Note that we can also alternatively express optimization (1) using the standard Dynamic Program (DP) formulation. The DP formulation is particularly useful for the purpose of studying the structure of the optimal policy of (1). However, since we will not be using this formulation in our analysis, we do not present it here. We refer readers who are interested in the DP formulation of optimization (1) to some papers in the literature (see, e.g., Federgruen and Heching (1999) and the papers listed in Chen et al. (2021b)).

Asymptotic analysis. Motivated by the asymptotic analysis in the RM literature, our analysis in this paper will also focus on the setting where the mean demand is sufficiently large, which is useful for analytical tractability. Specifically, for concreteness, we will assume the following:

Assumption 2. *$T(h+b) \geq 1$ and $(h+b)(T+1)^3 m^{1/2} \exp\left\{-\frac{1}{18} m^{2(1-\theta)}\right\} \leq 1$.*

The condition $T(h+b) \geq 1$ is quite mild as we anticipate that, in practice, the value of either T or b is not too small. As for the second condition, recall that we are interested in studying the performance of the best FP policy in a setting where both the mean demand and lead time could simultaneously be large. One way to do this is to directly express L and T as increasing functions of m. For example, we can set $T = m^{n'}$ and $L = m^n$ for some $n' > n$. For this instance, as long as m is sufficiently large, the second condition in Assumption 2 is immediately satisfied. In general, the second condition in Assumption 2 allows both L and T to grow up to exponentially in m. Thus, as long as m is sufficiently large, both L and T can also be quite large. For the purpose of our discussions in this paper, we will not need to explicitly express L and T as functions of m since our bounds already explicitly depend on these variables.

2.3. Additional notations

Let $r(p) := p\lambda(p) = p \cdot (1-p)$ denote the revenue rate under price p. To express the direct dependency of revenue rate on purchase rate instead of price, by abuse of notation, we will often write $r(\lambda) := \lambda \mathbf{P}(\lambda) = \lambda \cdot (1-\lambda)$, where $p(\lambda) = 1-\lambda$ is the inverse of $\lambda(\cdot)$. We will also use v^π to denote the expected profit under policy π.

3. Statements of Results and Discussions

In this section, we present the formal statements of our results and discuss their implications. The analysis of these results can be found in the next two sections. For brevity, throughout the remainder of this paper, we will simply denote the best FP policy as BFP.

3.1. Statement of Results

Our results in this paper are presented below.

A lower bound of the optimal expected profit (Theorem 1 in Section 4). Suppose that Assumptions 1 and 2 hold. There exists a policy $\bar{\pi}$ such that

$$v^* \geq v^{\bar{\pi}} \geq \frac{3}{16} Tm - (h+b)Tm^\theta - 2. \tag{2}$$

The limitation of FP policy (Theorem 2 in Section 5). Suppose that Assumptions 1 and 2 hold. Then, the expected profit under BFP is of order

$$O\left(m \cdot \left(\log T + \frac{m^{2(1-\theta)}}{\xi^2} + T \exp\left\{-\frac{\xi^2}{m^{2(1-\theta)}} L\right\}\right)\right), \tag{3}$$

where $\xi = h\Phi^{-1}(b/(h+b))/2$. Moreover, the constants hidden inside the Big-Oh notation are independent of all problem parameters (i.e., independent of m, L, T, θ, h, and b).

3.2. Discussions

Lower bound for the optimal expected profit. We start by discussing the lower bound in (2). Note that this bound is linear in m, which is not surprising since the mean market size in each period is m. In particular, the bound is independent of L and is linear in T, which tells us that the optimal expected profit is *linear* in the length of the selling horizon. The proof of this result utilizes a construction of a feasible policy $\bar{\pi}$ that places a constant order quantity in each period and applies a simple dynamic pricing policy to carefully control the impact of historical demand uncertainties on future inventory trajectory (in terms of ordering policy, $\bar{\pi}$ shares the same spirit as the policies analyzed in Chen et al. (2023b) and Liang et al. (2020)). We do not claim that $\bar{\pi}$ is near-optimal. Its construction merely serves the purpose of showing that the expected profit under the optimal joint inventory and pricing policy is linear in the length of the selling horizon, which is useful for assessing the value of dynamic pricing.

Upper bound for BFP. We now discuss the upper bound for the expected profit under BFP. Note that, when L is very small, the bound is linear in T; as L grows, the bound decays *exponentially* in L; and, eventually, when $L > (m^{2(1-\theta)}/\xi^2) \cdot \log T$, the bound becomes independent of L. In particular, for all $L > (m^{2(1-\theta)}/\xi^2) \cdot \log T$ (i.e., sufficiently large L), the bound is roughly of order $m \cdot (\log T + m^{2(1-\theta)}/\xi^2)$, which only grows *logarithmically* in T and is in stark contrast with the lower bound for the optimal profit in (2), which is linear in both m and T.

The bound in (3) also reveals an interesting insight regarding the impact of demand variability on expected profit. Specifically, the larger the value of θ, the smaller the term $m^{2(1-\theta)}/\xi^2$ in the bound and the larger the decay rate. This means that the expected profit under BFP becomes more sensitive to the length of lead time as demand variability grows (as long as $\theta < 1$).

Finally, it is worth noting here that holding cost is the main reason why BFP only has a relatively small expected profit that grows logarithmically in T for large L even in the setting where demand variability is smaller than its mean. If $h = 0$ (i.e., there is no holding cost), it is not difficult to construct a feasible FP policy that guarantees an expected profit that is linear both in m and T. For example, consider a policy that sets $p_t = 1/2$ (which is the unconstrained optimizer of $r(p)$) for all t, $Q_1 = m/2 + m^\theta\sqrt{T \log mT}$, and $Q_t = m/2$ for all $t > 1$. We can view $m^\theta\sqrt{T \log mT}$ as the safety stock for the system, which is more than sufficient to guarantee that stock-out will never happen throughout the horizon with a high probability. It is not difficult to see that this policy is asymptotically optimal with an expected loss at most of order $\sqrt{mT}$, similar to the performance of FP policy in the RM literature. If $h > 0$, however, the performance of the best FP policy becomes sensitive to lead time. This is so because, even when demand variability is small (i.e., $\theta < 1$), the system still needs to carry enough safety stock to hedge against cumulative demand uncertainties over L periods, which potentially incurs a large cumulative holding cost. As a consequence, as L keeps growing, the cumulative holding cost might grow too large to the point that it is no longer profitable to set the prices in a way that generates demand across all periods. Instead, it might be better to simply generate most of the demands in the last few periods and reduce demand in earlier periods by setting high prices during these periods. This strategy allows us to cut down the amount of needed safety stock, which in turn reduces the total holding costs. In Section 5, we show that this is indeed the property of the optimal fixed prices under our proxy model (see the monotonicity of purchase rates in Lemma 3 and Figure 1).

4. A Lower Bound for the Optimal Policy

In this section, we derive a lower bound for v^*. We do this by introducing a feasible policy $\bar{\pi}$ and derive a lower bound for $v^{\bar{\pi}}$. Since $v^* \geq v^{\bar{\pi}}$, a lower bound of $v^{\bar{\pi}}$ is immediately a lower bound of v^*. As noted earlier in the introduction, we do *not* claim that the policy $\bar{\pi}$ is near-optimal. Its construction only serves the purpose of showing that the expected profit under the optimal joint inventory and pricing policy is *linear* in the length of the selling horizon.

4.1. The definition of $\bar{\pi}$

The definition of $\bar{\pi}$ is given below.

Policy $\bar{\pi}$

Let $\Delta_0 := 0$, $\delta_{-1} := 0$, $\delta_0 := 1$. For each $t = 1, \ldots, T$, do:

1. Set $Q_t = \frac{m}{2}$
2. Calculate $\Delta_{t-1} := M_{t-1}(m, \theta) - m$
3. Calculate $\delta_{t-1} = 1 - \mathrm{Proj}_{[-\frac{1}{3}, \frac{1}{3}]}\left(\frac{\Delta_{t-1}}{m}\right)\delta_{t-2}$
4. Set $\lambda_t = \frac{1}{2}\delta_{t-1}$ and $p_t = p(\lambda_t)$

Note that the order quantity Q_t is fixed at the beginning of the horizon, i.e., the replenishment policy under $\bar{\pi}$ is not adaptive. As for the variable Δ_{t-1}, recall that $\mathbf{E}[M_{t-1}(m, \theta)] = m$. Thus, Δ_{t-1} represents the randomness in market size in period $t-1$.

The intuition behind the design of $\bar{\pi}$ is as follows. If θ is very small (i.e., demand is essentially deterministic), then $\bar{\pi}$ yields $\lambda_t \approx \frac{1}{2}$, which is close to optimal since $\frac{1}{2}$ is the unconstrained optimizer of $r(\lambda)$. If, however, demand is random, prices have to be dynamically adjusted to minimize the impact of past randomness on future states of the system. Our update formula for λ_t is motivated by the following logic. We want to set λ_t such that the following equation is satisfied:

$$m\lambda_t \;=\; \frac{m}{2} - [D_{t-1}(\lambda_{t-1}, m, \theta) - m\lambda_{t-1}] \;=\; \frac{m}{2} - \Delta_{t-1}\lambda_{t-1}, \tag{4}$$

which yields the following formula:

$$\lambda_t \;=\; \frac{1}{2}\left(1 - \frac{2\Delta_{t-1}}{m}\lambda_{t-1}\right). \tag{5}$$

In (4), the excess (which could either be positive or negative) demand in the previous period is "corrected" in the current period. The projection operator in the definition of $\bar{\pi}$ is added to make sure that the update formula for λ_t is always feasible (i.e., $\lambda_t \in [0,1]$ for all t), since directly applying (5) may be infeasible. The feasibility of λ_t under $\bar{\pi}$ is not difficult to verify. Let $\epsilon_t = \mathrm{Proj}_{[-\frac{1}{3}, \frac{1}{3}]}\left(\frac{\Delta_t}{m}\right)$. For $t \geq 1$, we can alternatively express δ_t as follows:

$$\delta_t = 1 - \sum_{k=0}^{t-1}(-1)^k \cdot \left(\prod_{s=t-k}^{t} \epsilon_s\right).$$

Since $|\epsilon_t| \leq \frac{1}{3}$ for all t, by comparison with a geometric series, we must have $\delta_t \in [\frac{1}{2}, \frac{3}{2}]$ for all t. Thus, $\lambda_t \in [\frac{1}{4}, \frac{3}{4}] \subseteq [0,1]$ is feasible for all t.

4.2. A lower bound of $v^{\bar{\pi}}$

The following theorem gives us a lower bound for the expected profit under $\bar{\phi}$.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then,*

$$v^* \geq v^{\bar{\pi}} \geq \frac{3}{16}Tm - 3(h+b)Tm^{\theta}.$$

Note that the bound in Theorem 1 is *independent* of L. Since $\theta < 1$, the bound is essentially of order $\Theta(Tm)$, which is linear in both T and m.

5. An Upper Bound for the Best Fixed Price Policy

In this section, we derive an upper bound for the expected profit under BFP. Under an FP policy, the prices at all periods are decided simultaneously at the beginning of the horizon whereas the replenishment policy itself can be adaptive. Directly evaluating the expected profit under BFP is not easy since we need to optimize both the sequence of fixed prices and the adaptive replenishment policy. Thus, our approach is to derive an upper bound for v^{BFP} through a series of upper bounds.

Since $\lambda_t = \lambda(p_t) = 1 - p_t$ is invertible in p_t, we will directly use λ_t as the decision variable. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_T)$. In this section, we proceed as follows:

1. In subsection 5.1., we derive a deterministic upper bound for v^{BFP}, i.e., we show that

$$v^{BFP} \;\leq\; \max_{\vec{\lambda} \in [0,1]^T} H^{FP}(\vec{\lambda})$$

   for a deterministic function $H^{FP}(\vec{\lambda})$.

2. In subsection 5.2, we study the properties of the maximizer of $H^{FP}(\vec{\lambda})$, which we denote as $\vec{\lambda}^{FP}$. We show that either $\lambda_t^{FP} = 0$ for all t or $\lambda_t^{FP} > 0$ for all t and, moreover, we also have $\lambda_s^{FP} \leq \lambda_t^{FP}$ for all $s \leq t$ (i.e., $\vec{\lambda}^{FP}$ is monotonically non-decreasing over time).

3. In subsection 5.3, we bound $H^{FP}(\vec{\lambda})$ with a more analytically tractable function $\tilde{H}^{FP}(\vec{\lambda})$ using the monotonicity property of $\vec{\lambda}^{FP}$ derived in subsection 5.2.

4. Finally, in subsection 5.4, we further bound the maximum value of $\tilde{H}^{FP}(\vec{\lambda})$ by the maximum value of a univariate function $\hat{H}(\mu, L, T)$ and analytically derive an upper bound for $\max_{\mu \geq 0} \hat{H}(\mu, L, T)$. This is our final bound for v^{BFP}.

The complete chain of bounds that we use to derive our final bound for v^{BFP} is shown below:

$$v^{BFP} \;\leq\; \max_{\vec{\lambda} \in [0,1]^T} H^{FP}(\vec{\lambda}) \;\leq\; \max_{\vec{\lambda} \in [0,1]^T} \tilde{H}^{FP}(\vec{\lambda}) \;\leq\; \max_{\mu \geq 0}\; \hat{H}(\mu, L, T) \;\leq\; \text{our final bound.}$$

We now discuss the details.

5.1. Bounding v^{BFP} with $\max_{\vec{\lambda}\in[0,1]^T} H^{FP}(\vec{\lambda})$

Let $C^{FP}(\vec{\lambda})$ denote the minimum expected cost under $\vec{\lambda}$. That is,

$$\begin{aligned} C^{FP}(\vec{\lambda}) := \min\nolimits_{\pi\in\Pi} \ & \mathbf{E}\left[\sum_{t=1}^{T} G\left(I_t^\pi + X_{t-L}^\pi - D_t(\lambda_t, m, \theta)\right)\right] \\ \text{s.t.} \quad & I_{t+1}^\pi = I_t^\pi + X_{t-L}^\pi - D_t(\lambda_t, m, \theta) \quad \forall t, \\ & I_1^\pi = 0, \, X_t^\pi \geq 0 \quad \forall t, \end{aligned}$$

where, by abuse of notation, we use Π to denote the set of all feasible (non-anticipating) replenishment policies. Note that v^{BFP} can be expressed as:

$$v^{BFP} = \max\nolimits_{\vec{\lambda}\in[0,1]^T} \left\{\sum_{t=1}^{T} mr(\lambda_t) - C^{FP}(\vec{\lambda})\right\}.$$

Let $H_t^\pi(\vec{\lambda})$ denote the history of all realizations under π and $\vec{\lambda}$ up to the end of period t. Since unsatisfied demand in the current period is backlogged to the next period, we can express

$$I_t^\pi = \sum_{s=1}^{t-1} Q_s^\pi - \sum_{s=1}^{t-1} D_s(\lambda_s, m, \theta),$$

for each t. Thus, we can bound:

$$\begin{aligned} & \mathbf{E}\left[G(I^\pi + X_{t-L}^\pi - D_t(\lambda_t, m, \theta)) \;\middle|\; H_{t-L-1}^\pi(\vec{\lambda})\right] \\ & = \mathbf{E}\left[G\left(\sum_{s=1}^{t} Q_s^\pi - \sum_{s=1}^{t} D_s(\lambda_s, m, \theta)\right) \;\middle|\; H_{t-L-1}^\pi(\vec{\lambda})\right] \\ & \geq \min_{y_t\geq 0} \mathbf{E}\left[G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s, m, \theta)\right)\right]. \end{aligned} \tag{6}$$

Note that the minimization in (6) is the classic Newsvendor problem and the optimal y_t is given by the Newsvendor quantile solution. However, due to the projection operator in the definition of $D_t(\lambda_t, m, \theta)$, this quantile solution is not easy to characterize. Fortunately, under Assumption 2, we can "ignore" this projection operator and simply use the quantile solution for the unrestricted Normal random variable without too much lost in accuracy.

Let $\tilde{D}_s(\lambda_s, m, \theta) =^d N_t(m, m^\theta)\lambda_s$. The following lemma tells us the impact of ignoring the projection operator in the definition of $D_t(\lambda_t, m, \theta)$.

LEMMA 1. *For any $y_t \geq 0$, we have:*

$$\begin{aligned} & \mathbf{E}\left[G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s, m, \theta)\right) - G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s, m, \theta)\right)\right] \\ & \geq -5(h+b)(L+1)^{3/2} m^{1/2} \cdot \exp\left\{-\frac{1}{4} m^{2(1-\theta)}\right\}. \end{aligned}$$

By Lemma 1 and Assumption 2, since $L+1 \leq T$, we can bound:

$$C^{FP}(\vec{\lambda}) \geq -5 + \sum_{t=1}^{T} \min_{y_t \geq 0} \mathbf{E}\left[G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s, m, \theta) \right) \right].$$

Note that the cost of ignoring the projection operator in the definition of $D_t(\lambda_t, m, \theta)$ is only a constant. The solution to the minimization problem inside the above summation is given by the following Newsvendor quantile solution:

$$y_t^*(\vec{\lambda}) \;:=\; m \cdot \left(\sum_{s=\max\{1,t-L\}}^{t} \lambda_s \right) + m^\theta \cdot \left(\sum_{s=\max\{1,t-L\}}^{t} \lambda_s^2 \right)^{1/2} \cdot \Phi^{-1}\left(\frac{b}{h+b} \right),$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard Normal random variable.

Combining the above bound for $C^{FP}(\vec{\lambda})$ together with the fact that $G(x) \geq h \cdot x$, we get

$$v^{BFP} \leq \max_{\vec{\lambda} \in [0,1]^T} H^{FP}(\vec{\lambda}), \tag{7}$$

where $H^{FP}(\vec{\lambda})$ is defined as follows:

$$H^{FP}(\vec{\lambda}) \;:=\; \sum_{t=1}^{T} mr(\lambda_t) - hm^\theta \cdot \Phi^{-1}\left(\frac{b}{h+b} \right) \cdot \left[\sum_{t=1}^{T} \left(\sum_{s=\max\{1,t-L\}}^{t} \lambda_s^2 \right)^{1/2} \right].$$

It is not difficult to see that $H^{FP}(\vec{\lambda})$ is jointly concave in $\vec{\lambda}$. Let $\vec{\lambda}^{FP}$ denote the optimal solution of the maximization in (7). In the next subsection, we will derive useful properties of $\vec{\lambda}^{FP}$.

5.2. Properties of $\vec{\lambda}^{FP}$

We state two lemmas.

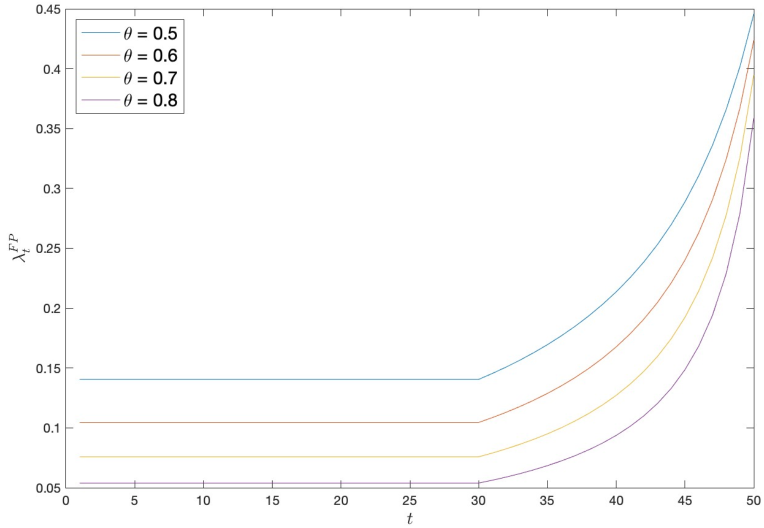
LEMMA 2. *Either $\lambda_s^{FP} = 0$ for all s or $\lambda_s^{FP} > 0$ for all s.*

The above lemma tells us that it is not possible to have an optimal solution $\vec{\lambda}^{FP}$ under which $\lambda_t^{FP} = 0$ for some t and $\lambda_t^{FP} > 0$ for others. The proof is by contradiction. We assume that there exists an optimal solution $\vec{\lambda}^{FP}$ such that $\lambda_t^{FP} = 0$ for some t and $\lambda_t^{FP} > 0$ for others, and show that we can construct an alternative solution $\vec{\lambda}$ that yields a larger value of $H^{FP}(\cdot)$, contradicting the optimality of $\vec{\lambda}^{FP}$. We defer the details to the Appendix.

Lemma 2 allows us to prove the following lemma.

LEMMA 3. *We have $0 \leq \lambda_1^{FP} \leq \lambda_2^{FP} \leq \cdots \leq \lambda_T^{FP} \leq 1$.*

Figure 1 The plots of $\vec{\lambda}^{FP}$ for the case where $T = 50$, $L = 20$, $m = 50$, $b = 4$, and $h = 2$. The values of λ_t^{FP} are monotonically non-decreasing over time.



Lemma 3 tells us that the optimal solution $\vec{\lambda}^{FP}$ is monotonically non-decreasing over time (see Figure 1 for a numerical illustration). This result is largely driven by the presence of holding cost. If $h = 0$, it is immediate from the definition of $H^{FP}(\vec{\lambda})$ that we must have $\lambda_1^{FP} = \lambda_2^{FP} = \cdots = \lambda_T^{FP}$. If, however, $h > 0$, it is better to generate more demand at later periods than at earlier periods to avoid incurring too much holding cost. Intuitively, since generating more demand requires us to carry more safety stock, which incurs more holding cost, generating more demand at later periods instead of earlier periods allows us to carry these extra safety stocks for a smaller number of periods, which results in smaller cumulative holding costs. The monotonicity result in Lemma 3 is useful for deriving a simple upper bound of $H^{FP}(\vec{\lambda})$, which we discuss in the next subsection.

It is worth noting here that, despite being intuitive, the proof of Lemma 3 is actually not straightforward. In the proof, we heavily utilize the well-known Karush-Kuhn-Tucker (KKT) conditions for optimality and proceed in the following two steps:

1. We first show that if there exists $\bar{s} \in \{2, 3, \ldots, T\}$ such that $\lambda_{\bar{s}-1}^{FP} > \lambda_{\bar{s}}^{FP}$, then we must have $\bar{s} \leq T - L$. In particular, if $T = L + 1$, this result immediately proves the lemma since the monotonicity of $\vec{\lambda}^{FP}$ is immediately guaranteed.

2. Given the above observation, we then prove the lemma by contradiction. Suppose that $\vec{\lambda}^{FP} \neq \vec{0}$ and there exists $\hat{t} \in \{2, 3, \ldots, T - L\}$ such that $\lambda_{\hat{t}-1}^{FP} > \lambda_{\hat{t}}^{FP}$. In such case, we show that we can always find some $\hat{t}_2 < \hat{t}_1$ such that some of the KKT conditions cannot simultaneously hold for both $\lambda_{\hat{t}_1}^{FP}$ and $\lambda_{\hat{t}_2}^{FP}$, contradicting the optimality of $\vec{\lambda}^{FP}$. Hence, $\vec{\lambda}^{FP}$ must be monotonic.

5.3. Bounding $\max_{\vec{\lambda}\in[0,1]^T} H^{FP}(\vec{\lambda})$ with $\max_{\vec{\lambda}\in[0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$

By Lemma 2, either $\lambda_s^{FP} = 0$ for all s or $\lambda_s^{FP} > 0$ for all s. Since the first scenario gives a trivial bound (i.e., $H^{FP}(\vec{0}) = 0$), in the remaining of this section, we will assume that $\lambda_s^{FP} > 0$ for all s. That is, we will focus on computing an upper bound for $H^{FP}(\vec{\lambda})$ on the set $(0,1]^T$ instead of $[0,1]^T$.

The challenge in directly analyzing $\max_{\vec{\lambda}\in(0,1]^T} H^{FP}(\vec{\lambda})$ is due to the “sum of square roots” term in the definition of $H^{FP}(\vec{\lambda})$. Thus, our idea is to bound $H^{FP}(\vec{\lambda})$ with another function that only contains a single square root term. The key observation that makes this reduction possible is stated in the following lemma.

LEMMA 4. *Suppose that* $0 \le x_1 \le x_2 \le \cdots \le x_T$. *Then,*

$$\sum_{t=1}^{T} \sqrt{x_t} \;\ge\; \left(\sum_{t=1}^{T} (1 + 2(T-t)) x_t \right)^{1/2}.$$

Proof. Simply square both sides and use the fact that $\sqrt{x_t} \cdot \left(\sqrt{x_{t+1}} + \cdots + \sqrt{x_T} \right) \;\ge\; (T-t)x_t$ (i.e., due to the monotonicity of $\vec{x}$). □

Note that the lower bound in Lemma 4 holds because of the monotonicity of $\vec{x}$. This is the reason why we first proved the monotonicity of $\vec{\lambda}^{FP}$ in subsection 5.2. In particular, applying Lemma 4 to the sum of square roots in the definition of $H^{FP}(\vec{\lambda})$ for monotonic $\vec{\lambda}$ yields:

LEMMA 5. *Suppose that* $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_T$. *Then,*

$$\sum_{t=1}^{T} \left(\sum_{s=\max\{1,t-L\}}^{t} \lambda_s^2 \right)^{1/2} \;\ge\; \left(\sum_{t=1}^{T} \gamma_t \lambda_t^2 \right)^{1/2},$$

where $\gamma_t = (L+1)^2 + 2(L+1)(T-L-t)$ *for* $t \le T-L-1$ *and* $\gamma_t = (T-t+1)^2$ *for* $t \ge T-L$.

Thus, we can define:

$$\tilde{H}^{FP}(\vec{\lambda}) \;:=\; \sum_{t=1}^{T} mr(\lambda_t) - hm^{\theta} \cdot \Phi^{-1}\left(\frac{b}{h+b}\right) \cdot \left(\sum_{t=1}^{T} \gamma_t \lambda_t^2 \right)^{1/2},$$

and bound $\max_{\vec{\lambda}\in(0,1]^T} H^{FP}(\vec{\lambda})$ from above with $\max_{\vec{\lambda}\in(0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$.

The benefit of using the simpler upper bound $\tilde{H}^{FP}(\vec{\lambda})$ instead of the original $H^{FP}(\vec{\lambda})$ is that, using the standard first-order optimality condition, it can be shown that $\max_{\vec{\lambda}\in(0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$ can be further bounded from above by the maximum of a univariate function, which eventually leads to our final bound for v^{BFP}. We discuss this next.

5.4. Bounding $\max_{\vec{\lambda}\in(0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$ with $\max_{\mu\geq 0} \hat{H}(\mu, L, T)$

Note that $\max_{\vec{\lambda}\in(0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$ is a multi-dimensional optimization problem. Although it is easier to analyze than the original $\max_{\vec{\lambda}\in(0,1]^T} H^{FP}(\vec{\lambda})$, its objective value is still not too easy to directly analyze. To help us derive our final bound for v^{BFP}, we will now take one last step and further bound $\max_{\vec{\lambda}\in(0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$ with the maximum of a univariate function.

Let $\vec{\lambda}^*$ denote the optimal solution to $\max_{\vec{\lambda}\in(0,1]^T} \tilde{H}^{FP}(\vec{\lambda})$. It is not difficult to verify that $\vec{\lambda}^*$ must be an interior solution. (In particular, if $\lambda_t^* = 1$ for some t, replacing λ_t^* with $\lambda_t = 1-\varepsilon$ for a sufficiently small ϵ increases the value of the objective function.) As a result, the optimal solution must satisfy the following first-order-optimality condition: For all t, we have

$$m - 2m\lambda_t^* - 2\xi \cdot m^\theta \gamma_t \lambda_t^* \left(\sum_{t=1}^T \gamma_t (\lambda_t^*)^2\right)^{-1/2} = 0,$$

where $\xi := h\Phi^{-1}(b/(h+b))/2$.

Now, let μ^* be defined below:

$$\mu^* := \xi m^{\theta-1} \left(\sum_{t=1}^T \gamma_t (\lambda_t^*)^2\right)^{-1/2}. \tag{8}$$

Given μ^*, we can express λ_t^* as follows:

$$\lambda_t^* = \frac{1}{2} \cdot \frac{1}{1+\mu^*\gamma_t}.$$

Finally, using the above expression, we can bound $\tilde{H}^{FP}(\vec{\lambda}^*)$ as follows:

$$\begin{aligned}
\tilde{H}^{FP}(\vec{\lambda}^*) &\leq \max_{\mu\geq 0} \left\{ \sum_{t=1}^T m \cdot \frac{1}{2} \cdot \frac{1}{1+\mu\gamma_t} \cdot \left(1 - \frac{1}{2} \cdot \frac{1}{1+\mu\gamma_t}\right) - \frac{2m^{2\theta-1}\xi^2}{\mu} \right\} \\
&\leq \max_{\mu\geq 0} \left\{ \frac{m}{2} \cdot \left(\sum_{t=1}^T \frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu} \right) \right\} \\
&:= \max_{\mu\geq 0} \hat{H}(\mu, L, T) = \hat{H}^*(L, T).
\end{aligned}$$

The following lemma gives us a property of $\hat{H}^*(L,T)$.

LEMMA 6. *For any given T, the function $\hat{H}^*(L,T)$ is non-increasing in L.*

The above lemma tells us that $\hat{H}^*(L,T)$ is non-increasing in L. This is an ideal property that is expected to hold for any reasonable upper bound of v^{BFP} (i.e., since we anticipate that v^{BFP} will be non-increasing in L too).

All that remains now is to compute an upper bound for $\hat{H}^*(L,T)$. We do this by splitting the feasible set $\{\mu \geq 0\}$ into three cases:

- Case 1: $\mu \geq 1$,
- Case 2: $(k+1)^{-2} < \mu \leq k^{-2}$ for some $k \leq L$,
- Case 3: $(k+1)^{-2} < \mu \leq k^{-2}$ for some $k \geq L+1$,

and derive an upper bound for each case. We state a lemma.

LEMMA 7. *We have:*

- *For case 1, we can bound:*

$$\hat{H}(\mu, L, T) \leq m \cdot \left(\frac{\pi^2}{32m^{2(\theta-1)}\xi^2} + \frac{\log T}{(L+1)^2} \right) := \mathcal{H}_1(L,T).$$

- *For case 2, for any $k \in \{1, 2, \ldots, L\}$, we can bound:*

$$\hat{H}(\mu, L, T) \leq m \cdot \left(2 + \log T + \frac{\pi^2}{32m^{2(\theta-1)}\xi^2} \right) := \mathcal{H}_2(L,T).$$

- *For case 3, for any $k \geq L+1$, we can bound:*

$$\hat{H}(\mu, L, T) \ \leq m \cdot \left(\frac{\pi^2}{32m^{2(\theta-1)}\xi^2} + \frac{4T}{\exp\left(\xi^2 m^{2(\theta-1)} \cdot L\right)} \right) := \mathcal{H}_3(L,T).$$

Putting all the bounds in Lemma 7 together yields our final bound:

THEOREM 2. *Suppose that Assumptions 1 and 2 hold. Then,*

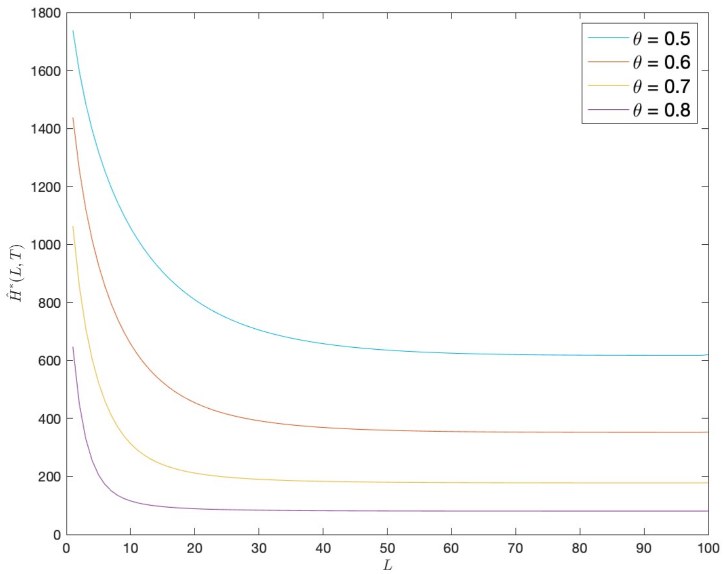
$$\begin{aligned} v^{BFP} \leq \hat{H}^*(L,T) &\leq \max\{\mathcal{H}_1(L,T), \mathcal{H}_2(L,T), \mathcal{H}_3(L,T)\} \\ &\leq m \cdot \left(2 + \log T + \frac{\pi^2}{32m^{2(\theta-1)}\xi^2} + \frac{4T}{\exp\left(\xi^2 m^{2(\theta-1)} \cdot L\right)} \right). \end{aligned}$$

The above bound highlights the non-trivial impact of lead time on the expected profit of BFP. When L is very small, the bound is linear in T; as L grows, the bound decays *exponentially* in L; and, eventually, when $L > (m^{2(1-\theta)}/\xi^2) \cdot \log T$, the bound becomes independent of L. (The exponential decay of the bound is consistent with the plots $\hat{H}^*(L,T)$ in Figure 2.) In particular, for all $L > (m^{2(1-\theta)}/\xi^2) \cdot \log T$, the bound is roughly of order $m \cdot (\log T + m^{2(1-\theta)}/\xi^2)$, which only grows logarithmically in T. This is in contrast to the bound in Theorem 1, which is linear in both m and t.

6. Conclusion

In this paper, we revisit the joint inventory and pricing problem with backlogging and positive lead time and address a fundamental question regarding the value of dynamic pricing. In contrast to most asymptotic analysis in the RM literature, which primarily focuses on the setting with a large mean demand, we allow the mean demand, the lead time, and the length of the selling horizon to

Figure 2 The plots of $\hat{H}^*(L,T)$ for the case where $T = 100$, $m = 50$, $b = 4$, and $h = 2$. As θ increases, $\hat{H}^*(L,T)$ drops faster and converges to a lower point.



simultaneously grow large. This is motivated by the fact that, in practice, mean demand is not the only parameter of interest and the values of these other parameters can also be large. Our results underscore the necessity of dynamic pricing, even in the setting where demand variability is (asymptotically) smaller than its mean, which is in contrast to the well-known result regarding the asymptotic optimality of the best FP policy in the RM literature.

References

Balseiro SR, Besbes O, Pizarro D (2023) Survey of dynamic resource-constrained reward collection problems: Unified model and analysis. *Operations Research* .

Bernstein F, Li Y, Shang K (2016) A simple heuristic for joint inventory and pricing models with lead time and backorders. *Management Science* 62(8):2358–2373.

Besbes O, Zeevi A (2009) Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms. *Operations research* 57(6):1407–1420.

Besbes O, Zeevi A (2012) Blind network revenue management. *Operations research* 60(6):1537–1550.

Besbes O, Zeevi A (2015) On the (surprising) sufficiency of linear models for dynamic pricing with demand learning. *Management Science* 61(4):723–739.

Chen Q, Jasin S, Duenyas I (2015) Real-time dynamic pricing with minimal and flexible price adjustment. *Manage. Sci.* 62(8):2437–2455.

Chen Q, Jasin S, Duenyas I (2019) Nonparametric self-adjusting control for joint learning and optimization of multiproduct pricing with finite resource capacity. *Mathematics of Operations Research* 44(2):601–631.

Chen Q, Jasin S, Duenyas I (2021a) Joint learning and optimization of multi-product pricing with finite resource capacity and unknown demand parameters. *Operations Research* 69(2):560–573.

Chen Q, Lei Y, Jasin S (2023a) Real-time spatial–intertemporal pricing and relocation in a ride-hailing network: Near-optimal policies and the value of dynamic pricing. *Operations Research* .

Chen X, Hu P, Hu Z (2021b) Joint pricing and inventory decisions. *Available at SSRN 3971881* .

Chen X, Stolyar A, Xin L (2023b) Asymptotic optimality of constant-order policies in joint pricing and inventory control models. *Mathematics of Operations Research (Forthcoming)* .

Den Boer AV (2015) Dynamic pricing and learning: historical origins, current research, and new directions. *Surveys in operations research and management science* 20(1):1–18.

Federgruen A, Heching A (1999) Combined pricing and inventory control under uncertainty. *Operations research* 47(3):454–475.

Gallego G, Van Ryzin G (1994) Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Manage. Sci.* 40(8):999–1020.

Gallego G, Van Ryzin G (1997) A multiproduct dynamic pricing problem and its applications to network yield management. *Oper. Res.* 45(1):24–41.

Jasin S (2014) Reoptimization and self-adjusting price control for network revenue management. *Operations Research* 62(5):1168–1178.

Karlin S (1958) Inventory models of the arrow-harris-marschak type with time lag. *Studies in the mathematical theory of inventory and production* .

Keskin NB, Zeevi A (2014) Dynamic pricing with an unknown demand model: Asymptotically optimal semi-myopic policies. *Operations research* 62(5):1142–1167.

Lei M, Liu S, Jasin S, Vakhutinsky A (2022) Joint inventory and pricing for a one-warehouse multistore problem: Spiraling phenomena, near optimal policies, and the value of dynamic pricing. *Operations Research* .

Lei Y, Jasin S (2020) Real-time dynamic pricing for revenue management with reusable resources, advance reservation, and deterministic service time requirements. *Operations Research* 68(3):676–685.

Lei Y, Jasin S, Sinha A (2018) Joint dynamic pricing and order fulfillment for e-commerce retailers. *Manufacturing & Service Operations Management* 20(2):269–284.

Liang AJ, Jasin S, Uichanco J (2020) Combining a smart pricing policy with a simple replenishment policy: Managing uncertainties in the presence of stochastic purchase returns. *Available at SSRN 3563847* .

This page is intentionally blank. Proper e-companion title page, with INFORMS branding and exact metadata of the main paper, will be produced by the INFORMS office when the issue is being assembled.

EC.1. Proof of Lemma 1

Fix t. Define G_t to be the event when the projection operator in the definition of $M_s(m,\theta)$ is not active from periods $\max\{1,t-L\}$ to t, i.e.,

$$\begin{aligned} G_t &:= \left\{M_s(m,\theta) = N_s(m,m^\theta),\ \forall s \in [\max\{1,t-L\},t]\right\} \\ &= \left\{0 \leq N_s(m,m^\theta) \leq 2m,\ \forall s \in [\max\{1,t-L\},t]\right\}. \end{aligned}$$

By the independent assumption, the standard tail bound for a Normal random variable and the sub-additive property of probability measure, we have:

$$\mathbf{P}(G_t^c) \leq \frac{2(L+1)}{(2\pi)^{1/2}m} \exp\left\{-\frac{1}{2}m^{2(1-\theta)}\right\}.$$

Now, let $\tilde{D}_t(\lambda_t,m,\theta) = N_t(m,\theta)\lambda_t$. For any y_t, we can bound:

$$\begin{aligned} &\left|\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta)\right)^+ - \left(y_t - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right)^+\right| \\ &\quad \leq \left|\sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta) - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right|. \end{aligned}$$

Similarly,

$$\begin{aligned} &\left|\left(\sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta) - y_t\right)^+ - \left(\sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta) - y_t\right)^+\right| \\ &\quad \leq \left|\sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta) - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right|. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbf{E}\left[G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta)\right) - G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right)\right] \\ &\quad \geq -(h+b)\cdot \mathbf{E}\left[\left|\sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta) - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right|\right] \\ &\quad = -(h+b)\cdot \mathbf{E}\left[\left|\sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s,m,\theta) - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right| \cdot \mathbf{1}\{G_t^c\}\right] \\ &\quad \geq -(h+b)\cdot \mathbf{E}\left[\left(2Lm + \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s,m,\theta)\right) \cdot \mathbf{1}\{G_t^c\}\right], \end{aligned}$$

where the last inequality holds since, by definition, $D_s(\lambda_s,m,\theta) \leq 2m$ for all s.

Let $N_t = N_t(m, m^\theta)$. By Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{s=\max\{1,t-L\}}^{t} N_s\lambda_s\right)\cdot \mathbf{1}\{G_t^c\}\right] &\le \mathbf{E}\left[\left(\sum_{s=\max\{1,t-L\}}^{t} N_s\lambda_s\right)^2\right]^{1/2}\cdot \mathbf{P}(G_t^c)^{1/2} \\
&\le (L+1)^{1/2}\mathbf{E}\left[\sum_{s=\max\{1,t-L\}}^{t} N_s^2\lambda_s^2\right]^{1/2}\cdot \mathbf{P}(G_t^c)^{1/2} \\
&\le (L+1)^{1/2}\mathbf{E}\left[\sum_{s=\max\{1,t-L\}}^{t} N_s^2\right]^{1/2}\cdot \mathbf{P}(G_t^c)^{1/2} \\
&\le (L+1)\cdot(m^2+m^{2\theta})^{1/2}\cdot \mathbf{P}(G_t^c)^{1/2} \\
&\le \sqrt{2}\cdot(L+1)\cdot m\cdot \mathbf{P}(G_t^c)^{1/2},
\end{aligned}$$

where the fourth inequality holds since $\mathbf{E}[N_s^2] = m^2 + m^{2\theta}$.

Put all things together, we have:

$$\begin{aligned}
&\mathbf{E}\left[G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} D_s(\lambda_s, m, \theta)\right) - G\left(y_t - \sum_{s=\max\{1,t-L\}}^{t} \tilde{D}_s(\lambda_s, m, \theta)\right)\right] \\
&\ge -(h+b)\cdot\left[2Lm + \sqrt{2}(L+1)m\right]\cdot \mathbf{P}(G_t^c)^{1/2} \\
&\ge -4(h+b)(L+1)m\cdot \mathbf{P}(G_t^c)^{1/2} \\
&\ge -5(h+b)(L+1)^{3/2}m^{1/2}\cdot \exp\left\{-\frac{1}{4}m^{2(1-\theta)}\right\}.
\end{aligned}$$

EC.2. Proof of Theorem 1

We first bound the expected total revenue. As argued before the statement of Theorem 1, $\lambda_t \in [\frac{1}{4}, \frac{3}{4}]$ for all t under $\bar{\pi}$. Thus,

$$\mathbf{E}\left[\sum_{t=1}^{T} mr(\lambda_t)\right] \ge Tmr\left(\frac{1}{4}\right) = \frac{3}{16}Tm. \tag{EC.1}$$

We now bound the expected total cost. For each t, define an event $A_t := \{|\Delta_t| \le \frac{m}{3}\}$. Also, let $A := \cap_{t=1}^{T} A_t$. We now bound $\mathbf{P}(A^c)$. First note that for any $Z \sim \mathbf{N}(0,1)$, we have

$$\mathbb{P}(Z > a) = \frac{1}{\sqrt{2\pi}}\int_{x=a}^{\infty} e^{-x^2/2}du < \frac{1}{\sqrt{2\pi}}\int_{x=a}^{\infty}\frac{x}{a}\cdot e^{-x^2/2}dx = \frac{1}{(2\pi)^{1/2}\cdot a\cdot e^{-a^2/2}}.$$

Also note that for any t, $|\Delta_t| \le \frac{m}{3}$ iff $|N_t(m, m^\theta) - m| \le \frac{m}{3}$. Therefore, by the independent assumption and the sub-additive property of probability measure, we have

$$\mathbf{P}(A^c) \le \sum_{t=1}^{T}\mathbf{P}(A_t^c) = T\cdot P\left(Z > \frac{m^{(1-\theta)}}{3}\right) \le \frac{3T}{(2\pi)^{1/2}\cdot m^{(1-\theta)}}\exp\left\{-\frac{1}{18}\cdot m^{2(1-\theta)}\right\}.$$

On event A, the projection operator in the definition of δ_t is not active for all t. By the identity (4), it is not difficult to see that

$$\sum_{s=1}^{t} D_s(\lambda_s, m, \theta) = \sum_{s=1}^{t} m\lambda_s + \sum_{s=1}^{t} \Delta_s \lambda_s \;=\; \frac{tm}{2} + \Delta_t \lambda_t.$$

This allows us to express I_{t+1} more compactly as follows:

$$I_{t+1} = \sum_{s=1}^{t} Q_s - \sum_{s=1}^{t} D_s(\lambda_s, m, \theta) \;=\; -\Delta_t \lambda_t.$$

Since $\lambda_t \leq \frac{3}{4} \leq 1$ on A, we can bound:

$$G(I_{t+1}) \leq (h+b)|\Delta_t|.$$

We now consider what happens on A^c. We have:

$$\begin{aligned} G(I_{t+1}) &= h(I_{t+1})^+ + b(-I_{t+1})^+ \\ &\leq h \cdot \sum_{s=1}^{t} Q_s + b \cdot \sum_{s=1}^{t} D_s(\lambda_s, m, \theta) \\ &\leq h \cdot \frac{tm}{2} + b \cdot 2tm \;\leq\; 2(h+b)tm, \end{aligned}$$

where the second inequality follows since $D_s(\lambda_s, m, \theta) \leq 2m\lambda_s \leq 2m$.

Now, put the bounds together, we can bound the expected total cost under $\bar{\pi}$ as follows:

$$\begin{aligned} \sum_{t=1}^{T} \mathbf{E}[G(I_{t+1})] &= \sum_{t=1}^{T} \mathbf{E}[G(I_{t+1}) \cdot \mathbf{1}\{A\}] + \sum_{t=1}^{T} \mathbf{E}[G(I_{t+1}) \cdot \mathbf{1}\{A^c\}] \\ &\leq \sum_{t=1}^{T} (h+b)\mathbf{E}\left[|\Delta_t| \cdot \mathbf{1}\{A\}\right] + \sum_{t=1}^{T} 2(h+b)tm\mathbf{P}(A^c) \\ &\leq (h+b) \cdot \sum_{t=1}^{T} \sqrt{\mathbf{E}\left[|\Delta_t|^2\right]} + \sum_{t=1}^{T} 2(h+b)tm\mathbf{P}(A^c) \\ &= (h+b) \cdot Tm^\theta + 2(h+b) \cdot (T+1)^2 \cdot m\mathbf{P}(A^c) \\ &\leq (h+b) \cdot Tm^\theta + \frac{3}{(2\pi)^{1/2}} \cdot m^{\theta-1/2} \cdot (h+b) \cdot (T+1)^3 \cdot m^{1/2} \cdot \exp\left\{-\frac{1}{18} \cdot m^{2(1-\theta)}\right\} \\ &\leq (h+b) \cdot Tm^\theta + 2m^{\theta-1/2} \\ &\leq 3(h+b) \cdot Tm^\theta \end{aligned} \tag{EC.2}$$

where the second inequality holds by Cauchy-Schwarz Inequality; the second equality holds by the definition of Δ_t and the last two inequalities hold by Assumption 2. Putting (EC.1) and (EC.2) into (1) under $\pi := \bar{\pi}$ yields the final bound.

EC.3. Proofs of Lemma 2 and Lemma 3

To facilitate our analysis, we will express $\max_{\vec{\lambda}\in[0,1]^T} H^{FP}(\vec{\lambda})$ as a minimization problem:

$$\Upsilon := \min \sum_{t=1}^{T}\left(\sum_{s=\max\{1,t-L\}}^{t}\lambda_s^2\right)^{1/2} + C\cdot\sum_{t=1}^{T}\lambda_t^2 - C\cdot\sum_{t=1}^{T}\lambda_t \qquad \text{(EC.3)}$$
$$\text{s.t.}\quad 0\le\lambda_s\le 1,\quad \forall s\in[T]$$

where $C = m^{1-\theta}[h\Phi^{-1}(\frac{b}{h+b})]^{-1} > 0$. We will use η_s^1 and η_s^2 to denote the dual variables for constraints $\lambda_s \ge 0$ and $\lambda_s \le 1$, respectively. Let $F_t(\vec{\lambda}) := (\sum_{s=\max\{1,t-L\}}^{t}\lambda_s^2)^{1/2} + C\cdot\lambda_t^2 - C\lambda_t$ and the objective function of (EC.3) can be written as $\sum_{t=1}^{T} F_t(\vec{\lambda})$.

Since strong duality holds for (EC.3), KKT conditions are necessary and sufficient to characterize optimal solutions. To facilitate our analysis, we start with listing the KKT conditions. The Lagrangian function is defined as follows:

$$\mathcal{L}(\vec{\lambda},\vec{\eta}^1,\vec{\eta}^2) := -\sum_{t=1}^{T}F_t(\vec{\lambda}) + \sum_{s=1}^{T}\eta_s^1\cdot\lambda_s + \sum_{s=1}^{T}\eta_s^2\cdot(1-\lambda_s) \qquad \text{(EC.4)}$$

The KKT conditions are as follows:

- Stationarity: For any $s\in[T]$, we have $0\in\partial\mathcal{L}$ (i.e., 0 belongs to subdifferential of $\mathcal{L}$ at $\vec{\lambda}$) where

$$\partial\mathcal{L} := -\sum_{t=1}^{T}\partial F_t(\vec{\lambda}) + \vec{\eta}^1 - \vec{\eta}^2.$$

  According to the definition of subgradient, when $F_t(\cdot)$ is differentiable for all t at $\vec{\lambda}$, the stationarity condition is equivalent to

$$-\sum_{t=1}^{T}\nabla F_t(\vec{\lambda}) + \vec{\eta}^1 - \vec{\eta}^2 = 0.$$

- Complementary Slackness condition: $\eta_s^1\cdot\lambda_s = 0$, $\eta_s^2\cdot(1-\lambda_s) = 0$ for each $s\in[T]$.
- Feasibility: We omit the primal feasibility conditions. For the dual feasibility, η_s^1 and η_s^2 are non-negative for all $s\in[T]$.

For analytical convenience, we define $\lambda_t := 0$ for any $t<0$ or $t>T$. Note that when $L=0$, the optimization problem (EC.3) is equivalent to

$$\Upsilon := \min\ (1-C)\cdot\sum_{t=1}^{T}\lambda_t + C\cdot\sum_{t=1}^{T}\lambda_t^2$$
$$\text{s.t.}\quad 0\le\lambda_s\le 1,\quad \forall s\in[T]$$

It is not difficult to see that, in this case, $\vec{\lambda}^{FP} = (\bar{\lambda},\bar{\lambda},\ldots,\bar{\lambda})$ for some $\bar{\lambda}\in[0,1]$. Obviously, $\vec{\lambda}^{FP}$ satisfies the properties in the two lemmas. Thus, our proof here focuses on the case $L>0$. For brevity, we will drop the superscript "FP" in $\vec{\lambda}^{FP}$ and simply use $\vec{\lambda}$ to denote optimal solution.

EC.3.1. Proof of Lemma 2

We prove the lemma via contradiction. Suppose to the contrary that there exists some $\bar{t}$ such that $\lambda_{\bar{t}} = 0$, but $\vec{\lambda} \neq 0$. We consider the following two scenarios:

- *Scenario 1*: $\lambda_{\bar{t}} = 0$ and $\lambda_{\bar{t}+1} > 0$ for some $\bar{t} \le T-1$.
- *Scenario 2*: $\lambda_{\bar{t}} > 0$ and $\lambda_{\bar{t}+1} = 0$ for some $\bar{t} \le T-1$.

We now show that neither scenarios could happen.

Scenario 1

Consider the following alternative feasible solution:

$$\lambda'_t = \begin{cases} \lambda_t & t \le \bar{t}-1 \\ v & t = \bar{t} \\ \sqrt{\lambda_{\bar{t}+1}^2 - v^2} & t = \bar{t}+1 \\ \lambda_t & t \ge \bar{t}+2 \end{cases}$$

for some $0 \le v \le \frac{\sqrt{3}}{2} \cdot \lambda_{\bar{t}+1}$. The value of v will be decided later. Let $\Pi(\vec{\lambda})$ and $\Pi(\vec{\lambda}')$ denote the objective values of (EC.3) under $\vec{\lambda}$ and $\vec{\lambda}'$, respectively. Then,

$$\begin{aligned}\Pi(\vec{\lambda}') - \Pi(\vec{\lambda}) = &\left[\sqrt{\sum_{s=\bar{t}-L}^{\bar{t}-1} \lambda_s^2 + v^2} + \sqrt{(\lambda_{\bar{t}+1}^2 - v^2) + \sum_{s=\bar{t}+2}^{\bar{t}+L+1} \lambda_t^2} - C \cdot \left(v + \sqrt{\lambda_{\bar{t}+1}^2 - v^2}\right)\right] \\ &- \left[\sqrt{\sum_{s=\bar{t}-L}^{\bar{t}-1} \lambda_s^2} + \sqrt{\lambda_{\bar{t}+1}^2 + \sum_{s=\bar{t}+2}^{\bar{t}+L+1} \lambda_s^2} - C \cdot \lambda_{\bar{t}+1}\right].\end{aligned}$$

Define a, b, and z as follows:

$$a := \sum_{s=\bar{t}-L}^{\bar{t}-1} \lambda_s^2, \quad b := \sum_{s=\bar{t}+2}^{\bar{t}+L+1} \lambda_s^2 \text{ and } z := v^2$$

Then, we can express:

$$\begin{aligned}\Pi(\vec{\lambda}') - \Pi(\vec{\lambda}) &= \left[\sqrt{a+z} + \sqrt{\lambda_{\bar{t}+1}^2 - z + b} - C \cdot \left(\sqrt{z} + \sqrt{\lambda_{\bar{t}+1}^2 - z}\right)\right] - \left[\sqrt{a} + \sqrt{\lambda_{\bar{t}+1}^2 + b} - C \cdot \lambda_{\bar{t}+1}\right] \\ &=: \Psi(z). \end{aligned} \tag{EC.5}$$

To show that $\vec{\lambda}$ cannot be an optimal solution, it is sufficient that we show $\Psi(z) < 0$ for some $z \in (0, \lambda_{\bar{t}+1}^2)$. We consider the following two cases: $a = 0$ and $a > 0$.

Case 1: $a = 0$. We first show that, if $a = 0$, then we must have $C > 1$; otherwise $\vec{\lambda}$ cannot be optimal in the first place. We state a claim.

CLAIM EC.1. *Suppose that* $C \leq 1$ *and* $\vec{\lambda}$ *is an optimal solution to* (EC.3). *Then for any* $\bar{t}$ *with* $\lambda_{\bar{t}+1} > 0$, *we have* $\sum_{s=\bar{t}-L}^{\bar{t}} \lambda_s^2 > 0$.

Given that $C > 1$, we now show that there must exists some $z \in (0, \lambda_{\bar{t}+1}^2)$ such that $\Psi(z) < 0$. Note that, when $a = 0$, for $z \geq 0$, we have:

$$\begin{aligned}\Psi(z) &\leq -(C-1)\sqrt{z} + C\left[\lambda_{\bar{t}+1} - \sqrt{\lambda_{\bar{t}+1}^2 - z}\,\right] \\ &= -(C-1)\sqrt{z} + C\frac{z}{\lambda_{\bar{t}+1} + \sqrt{\lambda_{\bar{t}+1}^2 - z}} \;=\; -(C-1)\sqrt{z}\left[1 - \frac{C}{C-1}\frac{\sqrt{z}}{\lambda_{\bar{t}+1} + \sqrt{\lambda_{\bar{t}+1}^2 - z}}\right].\end{aligned}$$

It is not difficult to see that the term inside the $[\cdot]$ after the last equality is positive for a sufficiently small z. This completes case 1.

Case 2: $a > 0$. When $a > 0$, for $z \geq 0$, we have:

$$\begin{aligned}\Psi(z) &\leq \left[\sqrt{a+z} - \sqrt{a}\,\right] - C\sqrt{z} + C\left[\lambda_{\bar{t}+1} - \sqrt{\lambda_{\bar{t}+1}^2 - z}\,\right] \\ &= -C\sqrt{z}\left[-\frac{1}{C}\frac{\sqrt{z}}{\sqrt{a+z} + \sqrt{a}} + 1 - \frac{\sqrt{z}}{\lambda_{\bar{t}+1} + \sqrt{\lambda_{\bar{t}+1}^2 - z}}\right].\end{aligned}$$

Again, the term inside the $[\cdot]$ after the last equality is positive for a sufficiently small z.

Summarizing the two cases, we conclude that scenario 1 cannot happen.

Scenario 2

Consider the following alternative feasible solution $\vec{\lambda}'$:

$$\lambda'_t = \begin{cases} \lambda_t & t \leq \bar{t} - 1 \\ \sqrt{\lambda_{\bar{t}}^2 - v^2} & t = \bar{t} \\ v & t = \bar{t} + 1 \\ \lambda_t & t \geq \bar{t} + 2 \end{cases}$$

for some $0 \leq v < \lambda_{\bar{t}}$. The value of v will be decided later. Let $\hat{\Pi}(\vec{\lambda})$ and $\hat{\Pi}(\vec{\lambda}')$ denote the objective values of (EC.3) under $\vec{\lambda}$ and $\vec{\lambda}'$, respectively. Then,

$$\begin{aligned}\hat{\Pi}(\vec{\lambda}') - \hat{\Pi}(\vec{\lambda}) &= \left[\sqrt{\hat{b} + \lambda_{\bar{t}}^2 - z} + \sqrt{z + \hat{a}} - C \cdot \left(\sqrt{z} + \sqrt{\lambda_{\bar{t}}^2 - z}\,\right)\right] - \left[\sqrt{\hat{b} + \lambda_{\bar{t}}^2} + \sqrt{\hat{a}} - C \cdot \lambda_{\bar{t}}\right] \\ &=: \hat{\Psi}(z) \end{aligned} \tag{EC.6}$$

where $\hat{b} := \sum_{s=\bar{t}-L}^{\bar{t}-1} \lambda_s^2$, $\hat{a} := \sum_{s=\bar{t}+2}^{\bar{t}+L+1} \lambda_s^2$, and $z := v^2$. Note that $\Psi(z)$ and $\hat{\Psi}(z)$ share a similar structure with slight changes of constants only: $a \to \hat{a}, b \to \hat{a}, \lambda_{\bar{t}+1} \to \lambda_{\bar{t}}$. By similar arguments as in scenario 1, we consider the following two cases:

- *Case 1.* $\hat{a}=0$.
- *Case 2.* $\hat{a}>0$.

For case 1, by similar arguments in proving Claim EC.1 , we can show that if $\hat{a}=0$, then $C>1$; otherwise, $\vec{\lambda}$ cannot be optimal. Thus, we can bound

$$\hat{\Psi}(z) \leq -(C-1)\sqrt{z}\left[1-\frac{C}{C-1}\frac{\sqrt{z}}{\lambda_{\bar{t}}+\sqrt{\lambda_{\bar{t}}^2-z}}\right].$$

It is not difficult to see that the term inside the $[\cdot]$ after the last equality is positive for a sufficiently small z. This completes case 1.

For case 2, when $\hat{a}>0$, for $z\geq 0$, we have:

$$\begin{aligned}\hat{\Psi}(z) &\leq \left[\sqrt{\hat{a}+z}-\sqrt{\hat{a}}\ \right]-C\sqrt{z}+C\left[\lambda_{\bar{t}}-\sqrt{\lambda_{\bar{t}}^2-z}\ \right] \\ &= -C\sqrt{z}\left[-\frac{1}{C}\frac{\sqrt{z}}{\sqrt{\hat{a}+z}+\sqrt{\hat{a}}}+1-\frac{\sqrt{z}}{\lambda_{\bar{t}}+\sqrt{\lambda_{\bar{t}}^2-z}}\right].\end{aligned}$$

Again, the term inside the $[\cdot]$ after the last equality is positive for a sufficiently small z. This completes case 2. We conclude that scenario 2 cannot happen.

Since neither of the two scenarios could happen, we conclude that if $\vec{\lambda}$ is an optimal solution, then either $\vec{\lambda}=(0,0,\ldots,0)$ or $\lambda_s>0$ for any $1\leq s\leq T$. The completes the proof for Lemma 2.

EC.3.1.1. Proof of Claim EC.1. We prove the claim by contradiction. Suppose that $C\leq 1$ and $\vec{\lambda}$ is an optimal solution to (EC.3) with $\lambda_{\bar{t}+1}>0$. Moreover, $\sum_{s=\bar{t}-L}^{\bar{t}}\lambda_s^2=0$. Under this $\vec{\lambda}$, the objective in (EC.3) can be written as

$$\begin{aligned}&\sum_{t=1}^{T}\left(\sum_{s=t-L}^{t}\lambda_s^2\right)^{1/2}+C\cdot\sum_{t=1}^{T}\lambda_t^2-C\cdot\sum_{t=1}^{T}\lambda_t \\ &=\sum_{t=\bar{t}+1}^{\bar{t}+L+1}\left(\sum_{s=t-L}^{t}\lambda_s^2\right)^{1/2}+C\cdot\lambda_{\bar{t}+1}^2-C\cdot\lambda_{\bar{t}+1} \\ &\quad+\sum_{t=1}^{\bar{t}}\left(\sum_{s=s=t-L}^{t}\lambda_s^2\right)^{1/2}+\sum_{t=\bar{t}+L+2}^{T}\left(\sum_{s=t-L}^{t}\lambda_s^2\right)^{1/2}+C\cdot\sum_{t\leq T,t\neq\bar{t}+1}\lambda_t^2-C\cdot\sum_{t\leq T,t\neq\bar{t}+1}\lambda_t\end{aligned}$$

For $x>0$, define $\Phi(x)$ as follows:

$$\begin{aligned}\Phi(x)=&\sum_{t=\bar{t}+1}^{\bar{t}+L+1}\left(\sum_{\substack{s\neq\bar{t}+1\\ t-L\leq s\leq t}}\lambda_s^2+x^2\right)^{1/2}+C\cdot x^2-C\cdot x \\ &+\sum_{t=1}^{\bar{t}}\left(\sum_{s=t-L}^{t}\lambda_s^2\right)^{1/2}+\sum_{t=\bar{t}+L+2}^{T}\left(\sum_{s=t-L}^{t}\lambda_s^2\right)^{1/2}+C\cdot\sum_{t\leq T,t\neq\bar{t}+1}\lambda_t^2-C\cdot\sum_{t\leq T,t\neq\bar{t}+1}\lambda_t\end{aligned}$$

Note that when $x > 0$, $\Phi(x)$ is differentiable and $\Phi'(x)$ is continuous. In addition, when $x = \lambda_{\bar{t}+1}$, we have:

$$\Phi'(\lambda_{\bar{t}+1}) = \sum_{t=\bar{t}+1}^{\bar{t}+L+1} \frac{\lambda_{\bar{t}+1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_t^2}} + 2C \cdot \lambda_{\bar{t}+1} - C.$$

Since $\sum_{s=\bar{t}-L}^{\bar{t}} \lambda_s^2 = 0$, we have $\lambda_s = 0$ for any $\bar{t} - L \le s \le \bar{t}$. Thus

$$\sum_{t=\bar{t}+1}^{\bar{t}+L+1} \frac{\lambda_{\bar{t}+1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_t^2}} \ge \frac{\lambda_{\bar{t}+1}}{\sqrt{\sum_{s=\bar{t}-L+1}^{\bar{t}+1} \lambda_t^2}} = \frac{\lambda_{\bar{t}+1}}{\lambda_{\bar{t}+1}} = 1,$$

which implies $\Phi'(\lambda_{\bar{t}+1}) \ge 1 - C + 2C\lambda_{\bar{t}+1} > 1 - C$. Note that, if $C \le 1$, $\Phi'(\lambda_{\bar{t}+1}) > 0$, which means that $\lambda_{\bar{t}+1}$ cannot be optimal.

EC.3.2. Proof of Lemma 3

Obviously, $\vec{\lambda} = 0$ satisfies the property. In what follows, we will consider the case $\vec{\lambda} \ne \vec{0}$. We prove the lemma using the KKT conditions. The stationarity conditions at $\vec{\lambda}$ can be written as

$$\frac{\partial \mathcal{L}}{\partial \lambda_k} = -\sum_{t=k}^{L+k} \frac{\lambda_k}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - 2C \cdot \lambda_k + C + \eta_k^1 - \eta_k^2 = 0, \qquad \forall 1 \le k \le T - L - 1$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_k} = -\sum_{t=k}^{T} \frac{\lambda_k}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - 2C \cdot \lambda_k + C + \eta_k^1 - \eta_k^2 = 0, \qquad \forall T - L \le k \le T$$

We state a claim.

Claim EC.2. *Suppose $\vec{\lambda}$ satisfies the KKT conditions and there exists $\bar{s}$ with $2 \le \bar{s} \le T$ such that $\lambda_{\bar{s}-1} > \lambda_{\bar{s}}$. Then, we must have $\bar{s} \le T - L$.*

We now prove the lemma by contradiction using Claim EC.2. Suppose that there exists $\hat{t}$ satisfying $2 \le \hat{t} \le T - L$ such that $\lambda_{\hat{t}-1} > \lambda_{\hat{t}}$. In what follows, we will show that if this happens, then we can always find some $\hat{t}_2 < \hat{t}_1$ such that the stationary conditions (in the KKT conditions) cannot simultaneously hold for both $\lambda_{\hat{t}_1}$ and $\lambda_{\hat{t}_2}$, contradicting the optimality of $\vec{\lambda}$. Before introducing $\hat{t}_1$ and $\hat{t}_2$, we first define a set $\mathcal{K}$ as follows:

$$\mathcal{K} := \{t \,|\, 2 \le t \le T - L,\ \lambda_{t-1} > \lambda_t\}.$$

Let $\hat{t}_1$ and $\hat{t}_2$ be as defined below:

- $\hat{t}_1 :=$ the smallest period in the set $\arg\min_{t \in \mathcal{K}} \lambda_t$.
- $\hat{t}_2 :=$ the largest period in the set $\arg\max_{1 \le t \le \hat{t}_1} \lambda_t$.

Intuitively, $\hat{t}_1$ is the smallest period in $\mathcal{K}$ under which λ_t attains the minimum value in $\mathcal{K}$ and $\hat{t}_2$ is the largest period $t \le \hat{t}_1$ that attains the maximum value of λ_t during periods 1 to $\hat{t}_1$. Some crucial properties of $\hat{t}_1$ and $\hat{t}_2$ are stated in the following claim.

Claim EC.3. *For $\hat{t}_1, \hat{t}_2$ defined above, we have*

1. $\hat{t}_2 < \hat{t}_1 \le T - L$ *and* $\lambda_{\hat{t}_2} > \lambda_{\hat{t}_1}$.
2. *For any t satisfying $\hat{t}_2 < t < \hat{t}_1$, we have* $\lambda_{\hat{t}_2} > \lambda_t > \lambda_{\hat{t}_1}$.
3. *For any $t > \hat{t}_1$, we have* $\lambda_t \ge \lambda_{\hat{t}_1}$.

We now consider the KKT conditions related to $\lambda_{\hat{t}_1}$ and $\lambda_{\hat{t}_2}$. Since $\vec{\lambda}$ satisfies the KKT conditions, by the complementary conditions and the fact that $1 \ge \lambda_{\hat{t}_2} > \lambda_{\hat{t}_1} \ge 0$, we have $\eta^2_{\hat{t}_1} = \eta^1_{\hat{t}_2} = 0$ and, thus, by the stationarity conditions, we have

$$\frac{\partial \mathcal{L}}{\partial \lambda_{\hat{t}_1}} = -\sum_{t=\hat{t}_1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - 2C \cdot \lambda_{\hat{t}_1} + C + \eta^1_{\hat{t}_1} = 0 \tag{EC.7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{\hat{t}_2}} = -\sum_{t=\hat{t}_2}^{\hat{t}_2+L} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - 2C \cdot \lambda_{\hat{t}_2} + C - \eta^2_{\hat{t}_2} = 0 \tag{EC.8}$$

By the dual feasibility conditions, these two equalities further imply that

$$\sum_{t=\hat{t}_2}^{\hat{t}_2+L} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} < \sum_{t=\hat{t}_1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}}. \tag{EC.9}$$

Next, we consider two scenarios:

- Scenario 1: $\hat{t}_2 + L < \hat{t}_1$
- Scenario 2: $\hat{t}_2 + L \ge \hat{t}_1$

For the first scenario, one key observation is that, for any t with $\hat{t}_2 - L \le t \le \hat{t}_2 + L < \hat{t}_1$, we have $\lambda_t \le \lambda_{\hat{t}_2}$ (by the definition of $\hat{t}_2$). Therefore,

$$\sum_{t=\hat{t}_2}^{\hat{t}_2+L} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \ge \sum_{t=\hat{t}_2}^{\hat{t}_2+L} \frac{1}{\sqrt{L+1}} = \sqrt{L+1}.$$

Also, for any t with $\hat{t}_2 < \hat{t}_1 - L \le t \le \hat{t}_1 + L$, we must have $\lambda_t \ge \lambda_{\hat{t}_1}$ (by the second and the third parts of Claim EC.3). Therefore,

$$\sum_{t=\hat{t}_1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \le \sum_{t=\hat{t}_1}^{\hat{t}_1+L} \frac{1}{\sqrt{L+1}} = \sqrt{L+1}.$$

The above two inequalities imply

$$\sum_{t=\hat{t}_2}^{\hat{t}_2+L} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \geq \sum_{t=\hat{t}_1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}}.$$

However, this contradicts inequality (EC.9). We conclude that the first scenario cannot happen.

For the second scenario, since $\hat{t}_2 < \hat{t}_1 \leq \hat{t}_2 + L$ (by the first part of Claim EC.3 and the definition of scenario 2), (EC.9) can be expressed as

$$\sum_{t=\hat{t}_2}^{\hat{t}_1-1} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} + \sum_{t=\hat{t}_1}^{\hat{t}_2+L} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} < \sum_{t=\hat{t}_1}^{\hat{t}_2+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} + \sum_{t=\hat{t}_2+L+1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}}.$$

Since $\lambda_{\hat{t}_2} > \lambda_{\hat{t}_1}$ (by the first part of Claim EC.3), this implies

$$\sum_{t=\hat{t}_2}^{\hat{t}_1-1} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} < \sum_{t=\hat{t}_2+L+1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}}. \tag{EC.10}$$

By the definition of $\hat{t}_2$, we have $\lambda_t \leq \lambda_{\hat{t}_2}$ for any $t \leq \hat{t}_1$. Moreover, by the last two parts of Claim EC.3, we have $\lambda_t \geq \lambda_{\hat{t}_1}$ for any $\hat{t}_2 \leq t \leq T$. Therefore,

$$\sum_{t=\hat{t}_2}^{\hat{t}_1-1} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \geq \frac{\hat{t}_1 - \hat{t}_2}{\sqrt{L+1}},$$

$$\sum_{t=\hat{t}_2+L+1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \leq \frac{\hat{t}_1 - \hat{t}_2}{\sqrt{L+1}},$$

which together imply

$$\sum_{t=\hat{t}_2}^{\hat{t}_1-1} \frac{\lambda_{\hat{t}_2}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \geq \sum_{t=\hat{t}_2+L+1}^{\hat{t}_1+L} \frac{\lambda_{\hat{t}_1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}},$$

contradicting (EC.10), as well as (EC.9) by implication. Thus, the second scenario cannot happen.

We conclude that there is no optimal $\vec{\lambda}$ under which there is some $\hat{t}$ with $2 \leq \hat{t} \leq T$ that satisfies $\lambda_{\hat{t}-1} > \lambda_{\hat{t}}$. This completes the proof for Lemma 3. □

EC.3.2.1. Proof of Claim EC.2. To prove $\bar{s} \leq T - L$, it is sufficient to show that $\lambda_{T-L} \leq \lambda_{T-L+1} \leq \lambda_{T-L+2} \leq \cdots \leq \lambda_T$. We prove this via contradiction. Suppose there exists some $\bar{t}$ satisfying $T - L + 1 \leq \bar{t} \leq T$ such that $\lambda_{\bar{t}-1} > \lambda_{\bar{t}}$. Since $1 \geq \lambda_{\bar{t}-1} > \lambda_{\bar{t}} \geq 0$, by the stationarity conditions,

$$\frac{\partial \mathcal{L}}{\partial \lambda_{\bar{t}-1}} = -\sum_{t=\bar{t}-1}^{T} \frac{\lambda_{\bar{t}-1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - 2C \cdot \lambda_{\bar{t}-1} + C - \eta_{\bar{t}-1}^2 = 0, \tag{EC.11}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{\bar{t}}} = -\sum_{t=\bar{t}}^{T} \frac{\lambda_{\bar{t}}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - 2C \cdot \lambda_{\bar{t}} + C + \eta_{\bar{t}}^1 = 0. \tag{EC.12}$$

By the dual feasibility conditions, (EC.11) and (EC.12) imply

$$\sum_{t=\bar{t}}^{T} \frac{\lambda_{\bar{t}}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} + 2C \cdot \lambda_{\bar{t}} \geq \frac{\lambda_{\bar{t}-1}}{\sqrt{\sum_{s=\bar{t}-L-1}^{\bar{t}-1} \lambda_s^2}} + \sum_{t=\bar{t}}^{T} \frac{\lambda_{\bar{t}-1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} + 2C \cdot \lambda_{\bar{t}-1}. \tag{EC.13}$$

Since $\lambda_{\bar{t}-1} > \lambda_{\bar{t}}$, (EC.13) implies

$$0 > \sum_{t=\bar{t}}^{T} \frac{\lambda_{\bar{t}}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} - \sum_{t=\bar{t}}^{T} \frac{\lambda_{\bar{t}-1}}{\sqrt{\sum_{s=t-L}^{t} \lambda_s^2}} \geq \frac{\lambda_{\bar{t}-1}}{\sqrt{\sum_{s=\bar{t}-L-1}^{\bar{t}-1} \lambda_s^2}} + 2C \cdot (\lambda_{\bar{t}-1} - \lambda_{\bar{t}}) > 0.$$

Since $0 > 0$ cannot happen, we conclude that (EC.13) cannot hold. Thus, $\bar{t}$ cannot exist. $\square$

EC.3.2.2. Proof of Claim EC.3. We prove the three parts separately.

Part 1. Note that $\hat{t}_1 \leq T - L$ holds by definition of $\mathcal{K}$. Since $\lambda_{\hat{t}_1 - 1} > \lambda_{\hat{t}_1}$, by the definition of $\hat{t}_2$, we must have $\hat{t}_2 \leq \hat{t}_1 - 1 < \hat{t}_1$, which implies $\lambda_{\hat{t}_2} \geq \lambda_{\hat{t}_1 - 1} > \lambda_{\hat{t}_1}$.

Part 2. $\lambda_{\hat{t}_2} > \lambda_t$ holds by the definition of $\hat{t}_2$. As for $\lambda_t > \lambda_{\hat{t}_1}$, suppose to the contrary that there exists t satisfying $\hat{t}_2 < t < \hat{t}_1$ such that $\lambda_t \leq \lambda_{\hat{t}_1}$. Without loss of generality, we assume that it is the smallest period satisfying this inequality. By the definition of t, we must have $\lambda_{t-1} > \lambda_{\hat{t}_1} \geq \lambda_t$, indicating that $t \in \mathcal{K}$. This further implies that $\lambda_t = \lambda_{t_1}$ since $\lambda_{\hat{t}_1} = \min_{t \in \mathcal{K}}\{\lambda_t\}$. However, $\lambda_t = \lambda_{\hat{t}_1} = \min_{t \in \mathcal{K}}\{\lambda_t\}$ contradicts with the 'minimum index' assumption on $\hat{t}_1$ (i.e., since $t < \hat{t}_1$). We conclude that $\lambda_t \leq \lambda_{\hat{t}_1}$ cannot happen.

Part 3. We prove this part by contradiction. Suppose that there exists t' satisfying $\hat{t}_1 + 1 \leq t' \leq T$ such that $\lambda_{t'} < \lambda_{\hat{t}_1}$ (without loss of generality, let $t' \geq \hat{t}_1 + 1$ be the smallest of such periods). By the definition of t', we have $\lambda_{t'-1} \geq \lambda_{\hat{t}_1} > \lambda_{t'}$. This implies $t' \in \mathcal{K}$ and, therefore, $\lambda_{t'} \geq \lambda_{\hat{t}_1}$ according to the definition of $\hat{t}_1$. But this contradicts $\lambda_{t'} < \lambda_{\hat{t}_1}$. We conclude that such t' cannot exist.

This completes the proof for Claim EC.3. $\square$

EC.4. Proof of Lemma 5

Using the notations in Lemma 4, let

$$\begin{aligned}
x_1 &= \lambda_1^2 \\
x_2 &= \lambda_1^2 + \lambda_2^2 \\
&\vdots \\
x_L &= \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_L^2 \\
x_{L+1} &= \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_L^2 + \lambda_{L+1}^2 \\
x_{L+2} &= \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_{L+1}^2 + \lambda_{L+2}^2
\end{aligned}$$

$$
\begin{aligned}
x_{L+3} &= \lambda_3^2 + \lambda_4^2 + \cdots + \lambda_{L+2}^2 + \lambda_{L+3}^2 \\
&\vdots \\
x_T &= \lambda_{T-L}^2 + \lambda_{T-L+1}^2 + \cdots + \lambda_{T-1}^2 + \lambda_T^2
\end{aligned}
$$

Since $\vec{\lambda}$ is monotonic, $\vec{x}$ is also monotonic. Let $\alpha_t = 1 + 2(T-t)$. By Lemma 4, we have:

$$
\sum_{t=1}^{T} \sqrt{x_t} \;\geq\; \left(\sum_{t=1}^{T} \alpha_t x_t\right)^{1/2} \;=\; \left[\sum_{t=1}^{T} \lambda_t \cdot \left(\sum_{s=t}^{\min\{t+L,T\}} \alpha_s\right)\right]^{1/2} \;=\; \left(\sum_{t=1}^{T} \lambda_t \gamma_t\right)^{1/2} \tag{EC.14}
$$

where $\gamma_t := \sum_{s=t}^{\min\{t+L,T\}} \alpha_s$. More specifically,

$$
\begin{aligned}
\gamma_T &= \alpha_T \\
&= 1 + 2 \cdot 0 = 1^2 \\
\gamma_{T-1} &= \alpha_{T-1} + \alpha_T \\
&= 2 + 2 \cdot 1 = 2^2 \\
&\vdots \\
\gamma_{T-L+1} &= \alpha_{T-L+1} + \alpha_{T-L+2} + \cdots + \alpha_{T-1} + \alpha_T \\
&= L + 2 \cdot (1 + \cdots + (L-1)) = L^2 \\
\gamma_{T-L} &= \alpha_{T-L} + \alpha_{T-L+1} + \cdots + \alpha_{T-1} + \alpha_T \\
&= (L+1) + 2 \cdot (1 + \cdots + L) = (L+1)^2 \\
\gamma_{T-L-1} &= \alpha_{T-L-1} + \alpha_{T-L} + \cdots + \alpha_{T-2} + \alpha_{T-1} \\
&= (L+1) + 2 \cdot (2 + \cdots + (L+1)) = (L+1)^2 + 2 \cdot (L+1) \cdot 1 \\
\gamma_{T-L-2} &= \alpha_{T-L-2} + \alpha_{T-L-1} + \cdots + \alpha_{T-3} + \alpha_{T-2} \\
&= (L+1) + 2 \cdot (3 + \cdots + (L+2)) = (L+1)^2 + 2 \cdot (L+1) \cdot 2 \\
\gamma_{T-L-3} &= \alpha_{T-L-3} + \alpha_{T-L-2} + \cdots + \alpha_{T-4} + \alpha_{T-3} \\
&= (L+1) + 2 \cdot (4 + \cdots + (L+3)) = (L+1)^2 + 2 \cdot (L+1) \cdot 3 \\
&\vdots \\
\gamma_1 &= \alpha_1 + \alpha_2 + \cdots + \alpha_L + \alpha_{L+1} \;=\; (L+1)^2 + 2 \cdot (L+1) \cdot (T-L-1)
\end{aligned}
$$

More compactly, we can write:

$$
\begin{aligned}
\gamma_t &= (L+1)^2 + 2 \cdot (L+1) \cdot (T-L-t) \quad \text{for } t \leq T-L-1 \\
\gamma_t &= (T-t+1)^2 \quad \text{for } t > T-L-1
\end{aligned}
$$

EC.5. Proof of Lemma 6

The proof of Lemma 6 proceeds in two steps. To facilitate our analysis, we let $\gamma_t(L)$ be the γ_t defined under L (we assume T is a given constant) and we regard any L satisfying $L < T-1$ as feasible L.

First we show that for each t, we have $\gamma_t(L)$ is non-decreasing in L. This property directly follows the definition of $\gamma_t(L)$. According to their definitions in Lemma 5, we have

$$\gamma_t(L) = \begin{cases} (L+1)\cdot[2(T-t+1)-(L+1)] & L+1 \leq T-t \\ (T-t+1)^2 & L+1 > T-t \end{cases}$$

Since $T \geq t+L+1$, $\gamma_t(L)$ is non-decreasing in L.

Next, we prove the statement in the lemma. Note that for any given $\mu > 0$ and two feasible lead times $L_1 < L_2$, we have

$$\hat{H}(\mu, L_1, T) \geq \hat{H}(\mu, L_2, T).$$

since $\gamma_t(L_1) \leq \gamma_t(L_2)$ for each t. Therefore, if μ_1^*, μ_2^* maximize $\hat{H}(\mu, L_1, T)$ and $\hat{H}(\mu, L_2, T)$ respectively, we have

$$\hat{H}(L_2, T) = \hat{H}(\mu_2^*, L_2, T) \leq \hat{H}(\mu_2^*, L_1, T) \leq \hat{H}(\mu_1^*, L_1, T) = \hat{H}^*(L_1, T)$$

This completes the proof for the lemma.

EC.6. Proof of Lemma 7

First note that

$$\begin{aligned}
\hat{H}(\mu, L, T) &= \frac{m}{2}\cdot\left(\sum_{t=1}^{T}\frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu}\right)\\
&= \frac{m}{2}\cdot\sum_{t=T-L}^{T}\frac{1}{1+\mu\gamma_t} + \frac{m}{2}\cdot\left(\sum_{t=1}^{T-L-1}\frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu}\right)\\
&= \frac{1}{\sqrt{\mu}}\cdot\frac{m}{2}\cdot\sum_{\ell=1}^{L+1}\frac{\sqrt{\mu}}{1+\mu\ell^2} + \frac{m}{2}\cdot\left(\sum_{t=1}^{T-L-1}\frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu}\right)\\
&\leq \frac{1}{\sqrt{\mu}}\cdot\frac{m}{2}\cdot\sum_{\ell=1}^{L+1}\int_{x=\ell-1}^{\ell}\frac{d\sqrt{\mu}x}{1+\mu x^2} + \frac{m}{2}\cdot\left(\sum_{t=1}^{T-L-1}\frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu}\right)\\
&= \frac{1}{\sqrt{\mu}}\cdot\frac{m}{2}\cdot\int_{x=0}^{L+1}\frac{d\sqrt{\mu}x}{1+\mu x^2} + \frac{m}{2}\cdot\left(\sum_{t=1}^{T-L-1}\frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu}\right)\\
&= \frac{1}{\sqrt{\mu}}\cdot\frac{m}{2}\cdot\int_{z=0}^{\sqrt{\mu}(L+1)}\frac{dz}{1+z^2} + \frac{m}{2}\cdot\left(\sum_{t=1}^{T-L-1}\frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\mu}} \cdot \frac{m}{2} \cdot \tan^{-1}(\sqrt{\mu}(L+1)) + \frac{m}{2} \cdot \left(\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu} \right) \\
&\leq \frac{1}{\sqrt{\mu}} \cdot \frac{m}{2} \cdot \frac{\pi}{2} + \frac{m}{2} \cdot \left(\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu} \right) \\
&= \frac{m}{2} \cdot \left(\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} - \frac{4m^{2(\theta-1)}\xi^2}{\mu} + \frac{\pi}{2} \cdot \frac{1}{\sqrt{\mu}} \right) \\
&= \frac{m}{2} \cdot \left(\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} - \frac{2m^{2(\theta-1)}\xi^2}{\mu} - \frac{2m^{2(\theta-1)}\xi^2}{\mu} + \frac{\pi}{2} \cdot \frac{1}{\sqrt{\mu}} \right) \\
&\leq \frac{m}{2} \cdot \left(\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} - \frac{2m^{2(\theta-1)}\xi^2}{\mu} + \frac{\pi^2}{32m^{2(\theta-1)}\xi^2} \right) \qquad \text{(EC.15)}
\end{aligned}$$

where the first inequality holds since $\frac{\sqrt{\mu}}{1+\mu x^2}$ decreases in x and the second inequality holds since $\tan^{-1}(x) \leq \pi/2$ for any $x > 0$, the last inequality holds by maximizing the last two terms in the parenthesis over μ. We analyze each of the three cases respectively.

Case 1: $\mu \geq 1$

In this case, we only need to bound the summation term. In particular, given the expression of γ_t, for the first $T-L-1$ terms we have:

$$\begin{aligned}
\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} &\leq \sum_{t=1}^{T-L-1} \frac{1}{\mu\gamma_t} \leq \sum_{t=1}^{T-L-1} \frac{1}{\gamma_t} \leq \int_0^{T-L-1} \frac{dx}{(L+1)^2 + 2(L+1)x} \\
&\leq \frac{\log((L+1)^2 + 2(T-L-1))}{2(L+1)^2} = \frac{\log(L^2+2T-1)}{2(L+1)^2}.
\end{aligned}$$

By (EC.15), we have

$$H(\mu, L, T) \ \leq \ \frac{m}{2} \cdot \left[\frac{\pi^2}{32m^{2(\theta-1)}\xi^2} + \frac{\log(3T^2)}{2(L+1)^2} \right] < m \cdot \left(\frac{\pi^2}{64m^{2(\theta-1)}\xi^2} + \frac{\log T}{(L+1)^2} \right).$$

Case 2: $(k+1)^{-2} < \mu \leq k^{-2}$ where $1 \leq k \leq L$

Similar to the previous case, we only need to bound the summation term. We have

$$\begin{aligned}
\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} &= \sum_{\ell=1}^{T-L-1} \frac{1}{1+\mu \cdot [(L+1)^2 + 2(L+1)\ell]} \\
&\leq (k+1)^2 \cdot \sum_{\ell=1}^{T-L-1} \frac{1}{(L+1)^2 + 2(L+1)\ell} \\
&\leq (k+1)^2 \cdot \frac{\log(3T^2)}{2(L+1)^2} \ \leq \ \frac{\log(3T^2)}{2}.
\end{aligned}$$

By (EC.15), we have

$$\begin{aligned}\hat{H}(\mu, L, T) &\leq \frac{m}{2} \cdot \left(\frac{\pi^2}{32 m^{2(\theta-1)} \xi^2} + \frac{\log(3T^2)}{2} \right) \\ &\leq m \cdot \left(2 + \log T + \frac{\pi^2}{64 m^{2(\theta-1)} \xi^2} \right)\end{aligned}$$

Case 3: $(k+1)^{-2} < \mu \leq k^{-2}$ and $\mu \in S$ where $k > L$

For this case, we have

$$\begin{aligned}\sum_{t=1}^{T-L-1} \frac{1}{1+\mu\gamma_t} &= \sum_{\ell=1}^{T-L-1} \frac{1}{1+\mu \cdot [(L+1)^2 + 2(L+1)\ell]} \\ &\leq \int_0^{T-L-1} \frac{dx}{1+\mu \cdot [(L+1)^2 + 2(L+1)x]} \\ &\leq \frac{\log(1+\mu \cdot [(L+1)^2 + 2(L+1)(T-L-1))}{2(L+1)\mu} \\ &= \frac{\log(1+\mu \cdot (L+1)(2T-L-1))}{2(L+1)\mu} \\ &\leq \frac{\log(1+k^{-2} \cdot (L+1)(2T-L-1))}{2(L+1)(k+1)^{-2}} \\ &\leq \frac{2\log(1+k^{-2} \cdot (L+1)(2T-L-1))}{(L+1)k^{-2}}.\end{aligned}$$

Therefore, by (EC.15), we have

$$\begin{aligned}\hat{H}(\mu, L, T) &\leq \frac{m}{2} \cdot \left(\frac{2\log(1+k^{-2} \cdot (L+1)(2T-L-1))}{(L+1)k^{-2}} - 2\xi^2 \cdot m^{2(\theta-1)} \cdot k^2 + \frac{\pi^2}{16 m^{2(\theta-1)} \xi^2} \right) \\ &= m \cdot \left(\frac{\log(1+k^{-2} \cdot (L+1)(2T-L-1))}{(L+1)k^{-2}} - \xi^2 \cdot m^{2(\theta-1)} k^2 + \frac{\pi^2}{32 m^{2(\theta-1)} \xi^2} \right) \qquad \text{(EC.16)}\end{aligned}$$

In order to further bound (EC.15), consider the following optimization problem:

$$\max_{y \geq (L+1)^2} y \cdot \left(\frac{\log\left(1 + \frac{4(L+1)T}{y}\right)}{(L+1)} - \xi^2 \cdot m^{2(\theta-1)} \right). \qquad \text{(EC.17)}$$

Note that the optimal objective value in (EC.17) can be negative. If it is negative, we can simply bound

$$\hat{H}(\mu, L, T) \leq \frac{m}{2} \cdot \frac{\pi^2}{32 m^{2(\theta-1)} \xi^2}.$$

We now focus on the case where the optimal objective value in (EC.17) is non-negative. Letting y^* denote the optimal solution to (EC.17), the following must hold (otherwise, the objective value will be negative):

$$y^* \leq \frac{4(L+1)}{\exp\left(\xi^2 m^{2(\theta-1)} \cdot (L+1)\right) - 1} \cdot T. \qquad \text{(EC.18)}$$

The first-order-optimality condition implies

$$\frac{4T\cdot(y^*)^{-1}}{1+4(L+1)T\cdot(y^*)^{-1}}=\frac{\log\left(1+\frac{4(L+1)T}{y^*}\right)}{(L+1)}-\xi^2 m^{2(\theta-1)} \tag{EC.19}$$

Substituting (EC.18) and (EC.19) back to (EC.17), we have

$$\sum_{t=1}^{T-L-1}\frac{1}{1+\mu\gamma_t}-\frac{\xi^2\cdot m^{2(\theta-1)}}{\mu}\leq\frac{4T}{1+4(L+1)T\cdot(y^*)^{-1}}$$

Therefore, by (EC.16), we have

$$\begin{aligned}\hat{H}(\mu,L,T)&\leq m\cdot\left(\frac{\pi^2}{64m^{2(\theta-1)}\xi^2}+\frac{4T}{1+4(L+1)T\cdot(y^*)^{-1}}\right)\\&\leq m\cdot\left(\frac{\pi^2}{64m^{2(\theta-1)}\xi^2}+\frac{4T}{\exp\left(\xi^2 m^{2(\theta-1)}\cdot L\right)}\right),\end{aligned}$$

where the last inequality again follows by (EC.18).